Parameterized hardness of coding and lattice problems

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Why codes and lattices?

- Fundamental objects in mathematics and computer science.

- Computational problems on such objects are basis of post-quantum cryptography.

NIST

PROJECTS/PROGRAMS

Post-Quantum Cryptography

Summary

Post-Quantum Cryptography (PQC) - An area of cryptography that researches and advances the use of quantum-resistant primitives, with the goal of keeping existing public key infrastructure intact in a future era of quantum computing. Intended to be secure against both quantum and classical computers and deployable without drastic changes to existing communication protocols and networks.

What is a (linear) code?

$C \subseteq \mathbb{F}_q^n$ is a linear code if it is a vector subspace of $\mathbb{F}_q^n$.

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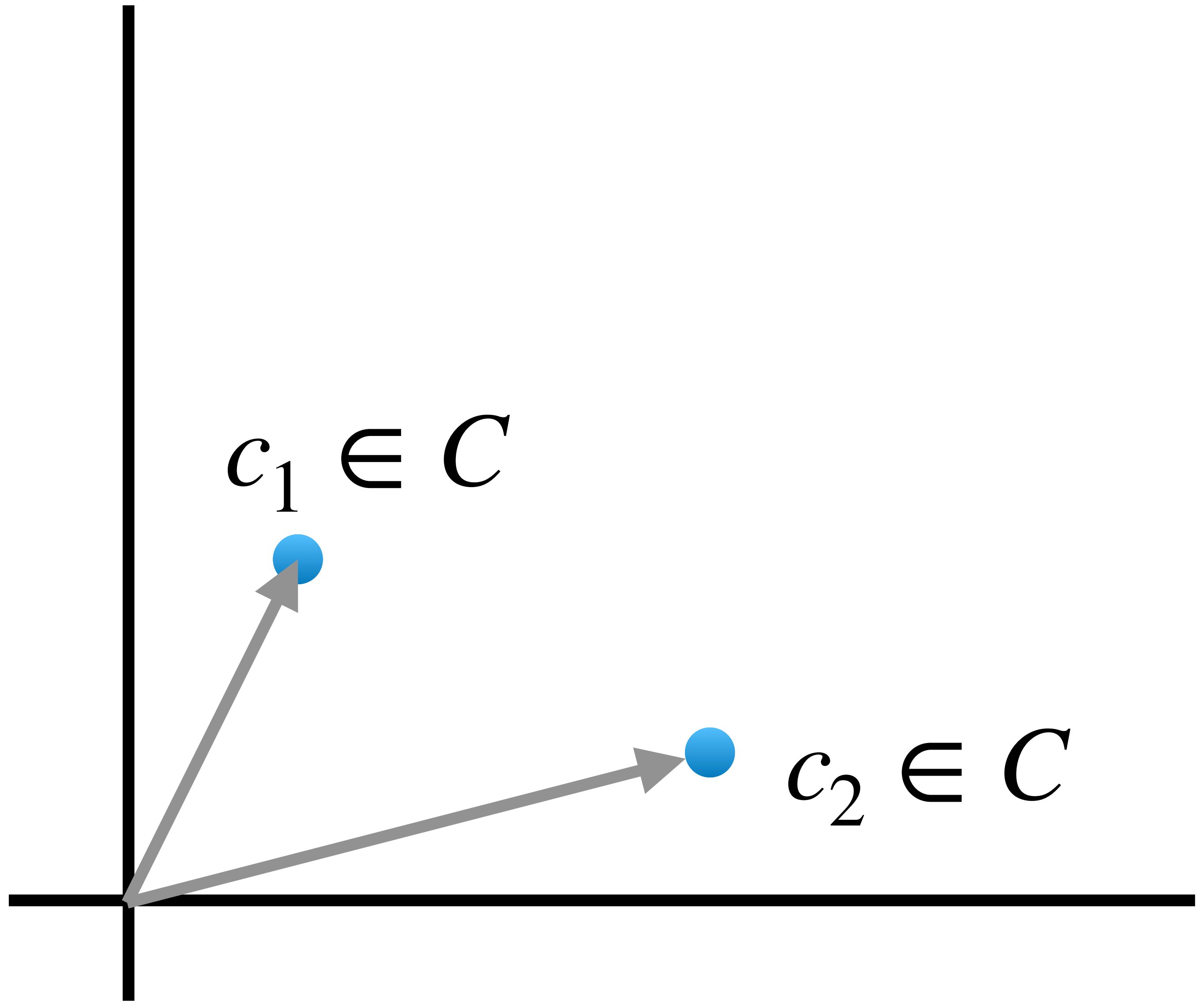

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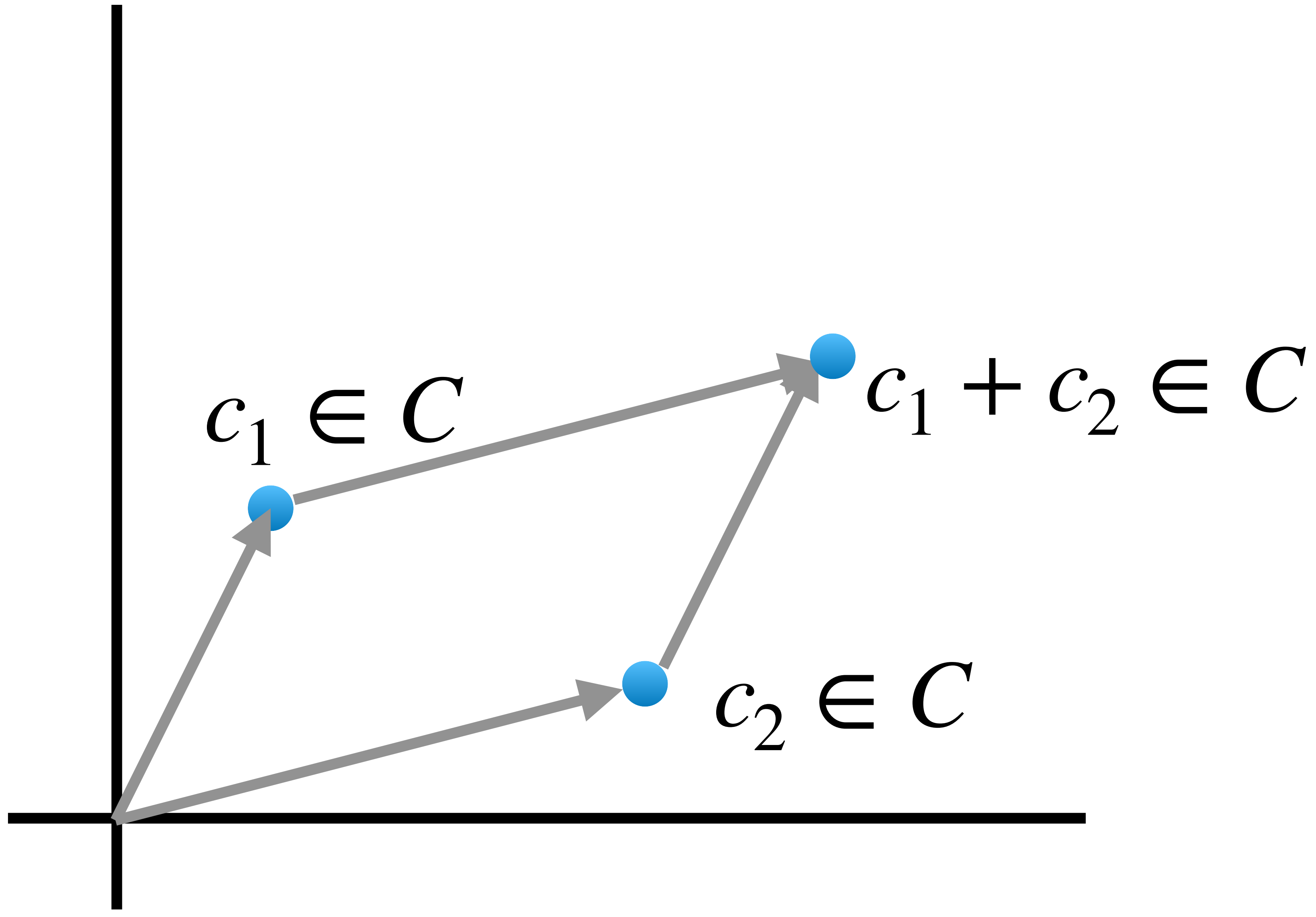

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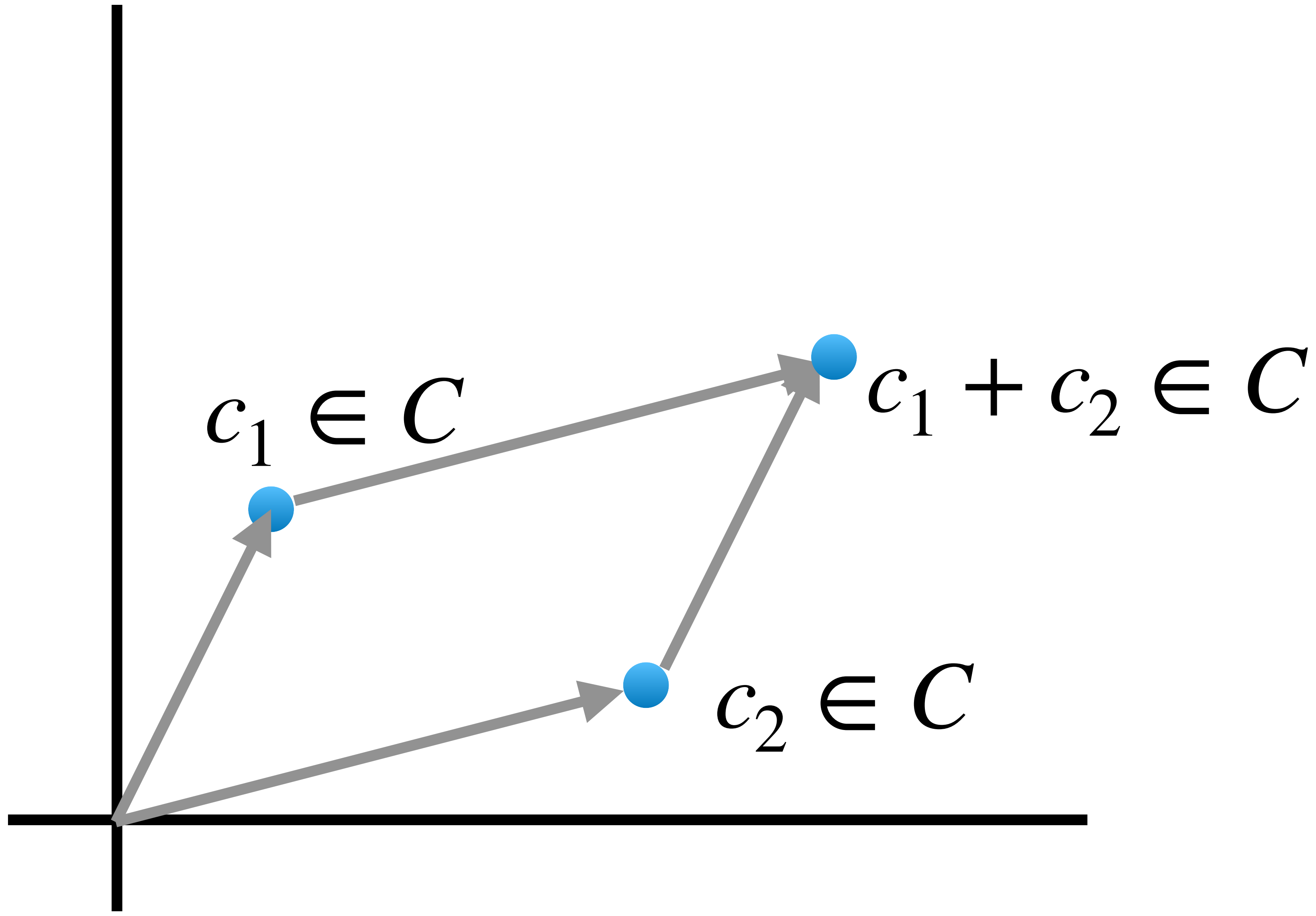

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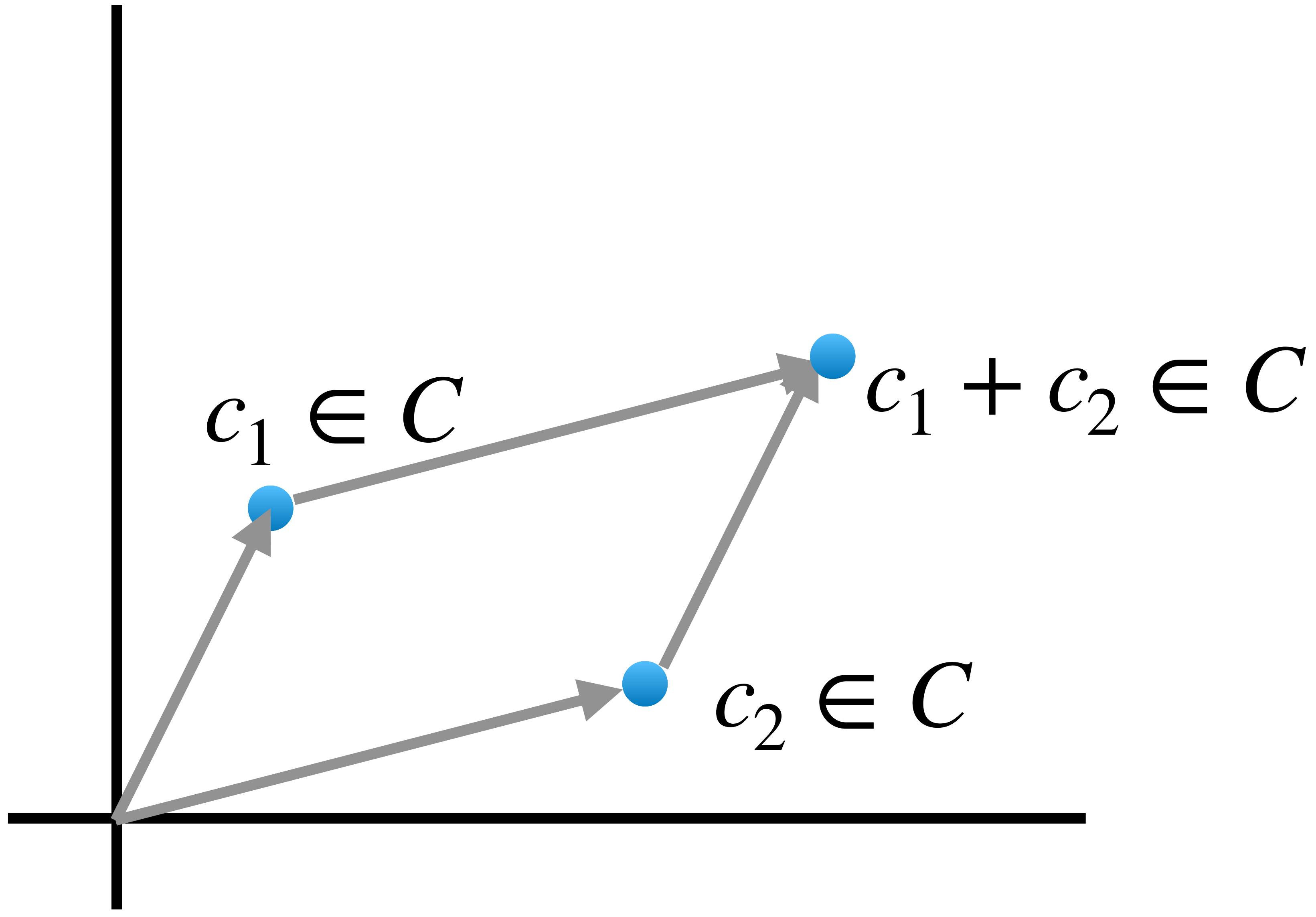

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$$d(C) = \min_{c,c' \in C, c \neq c'} \|c - c'\|_0$$

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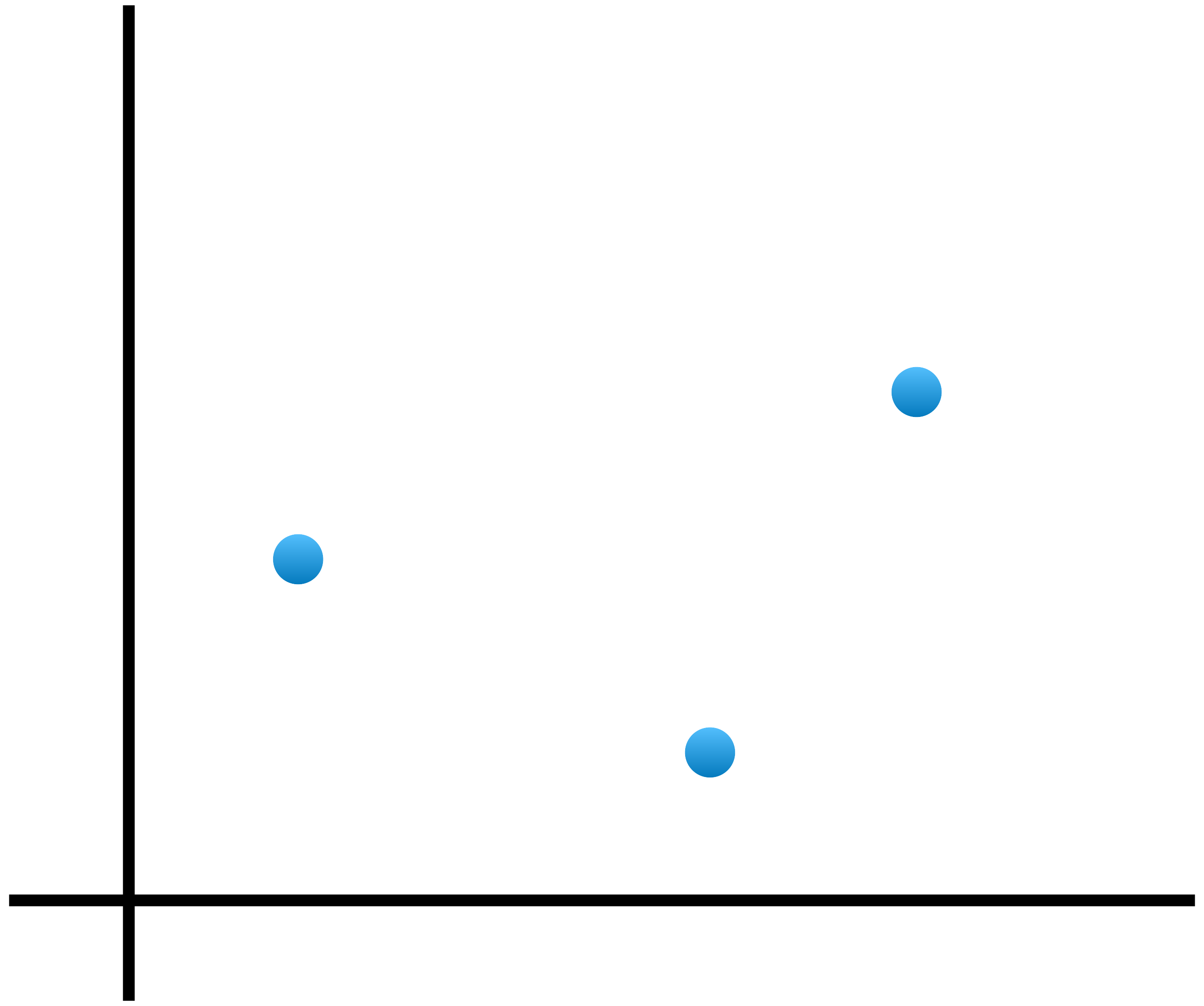

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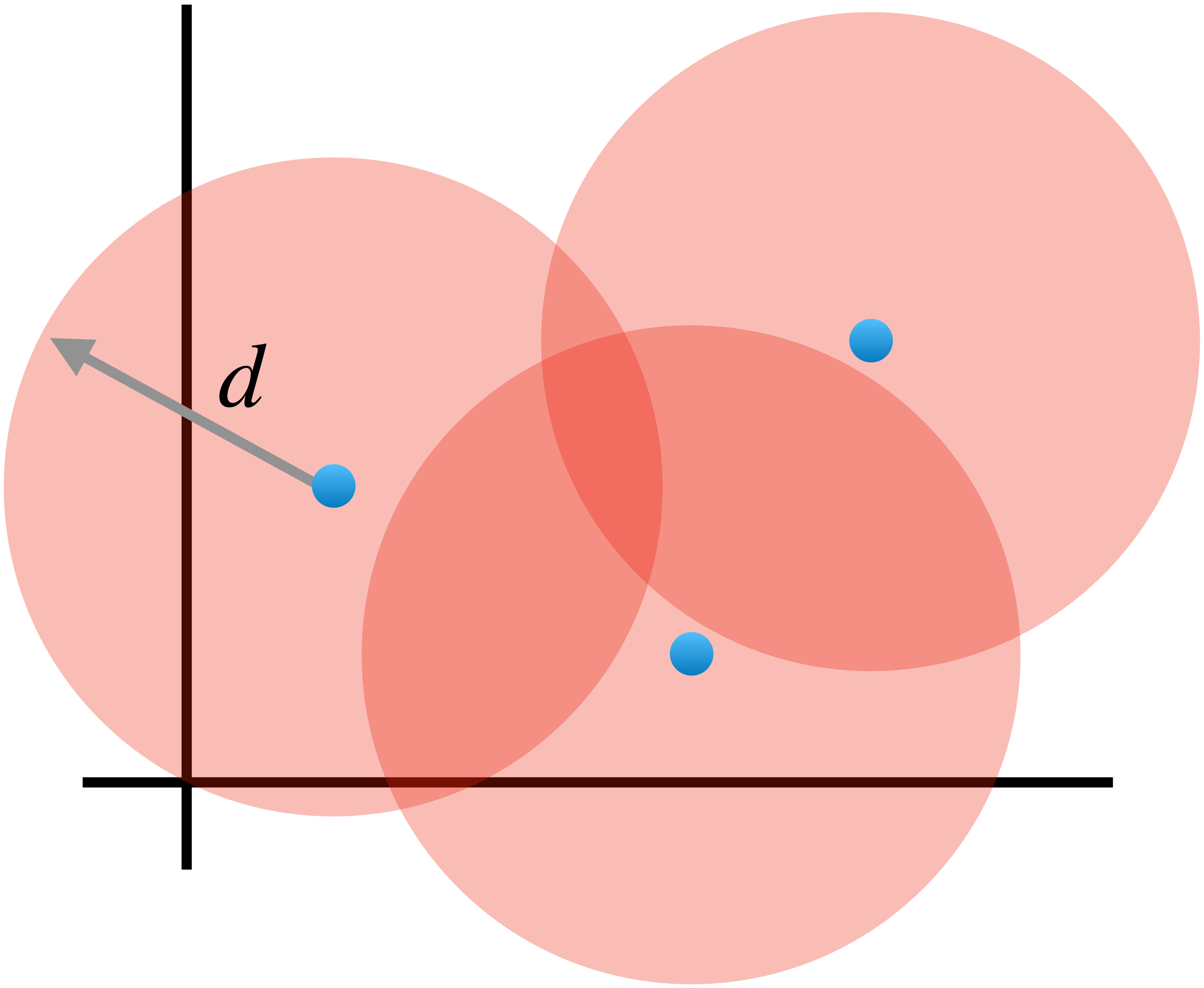

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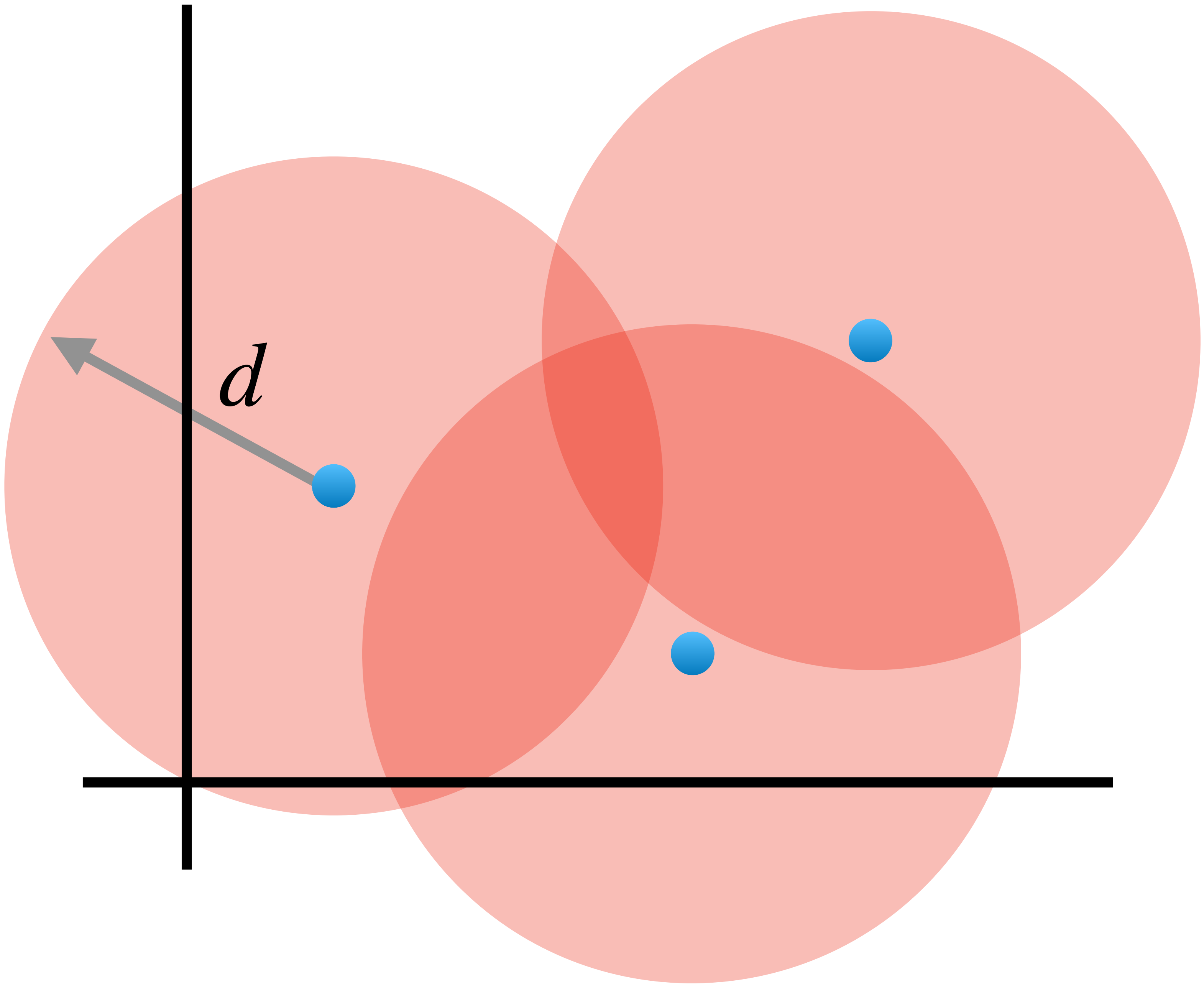

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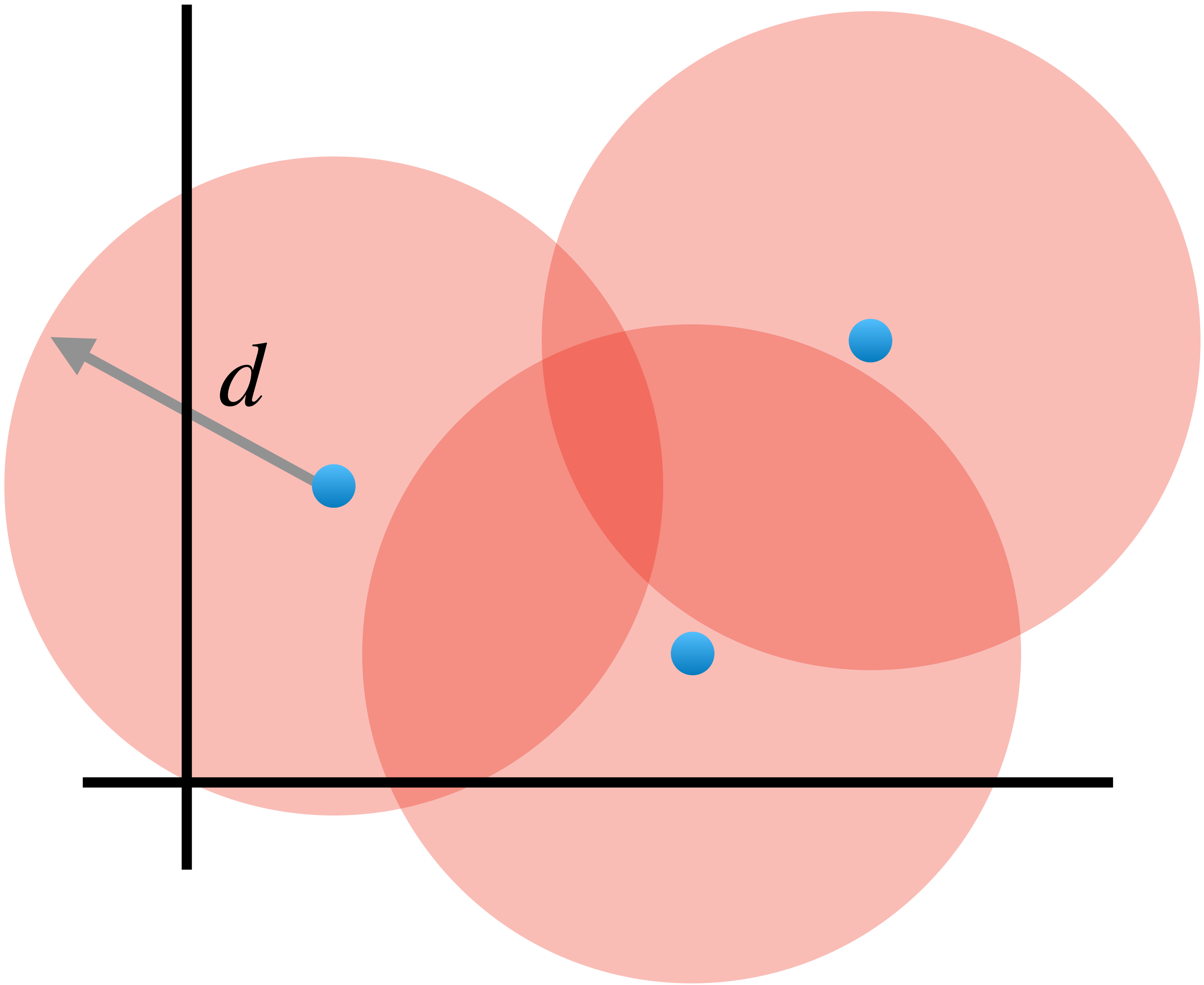

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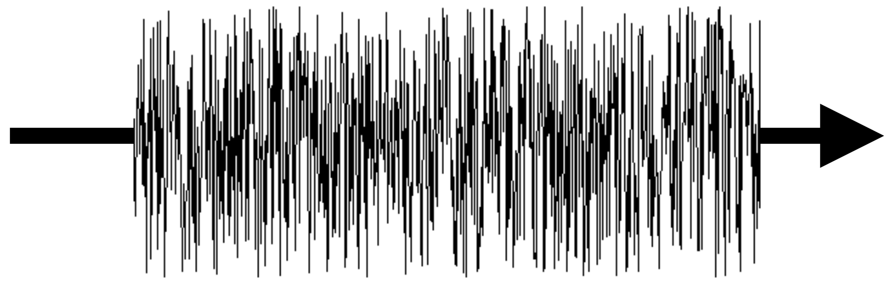
$c \in C$  z

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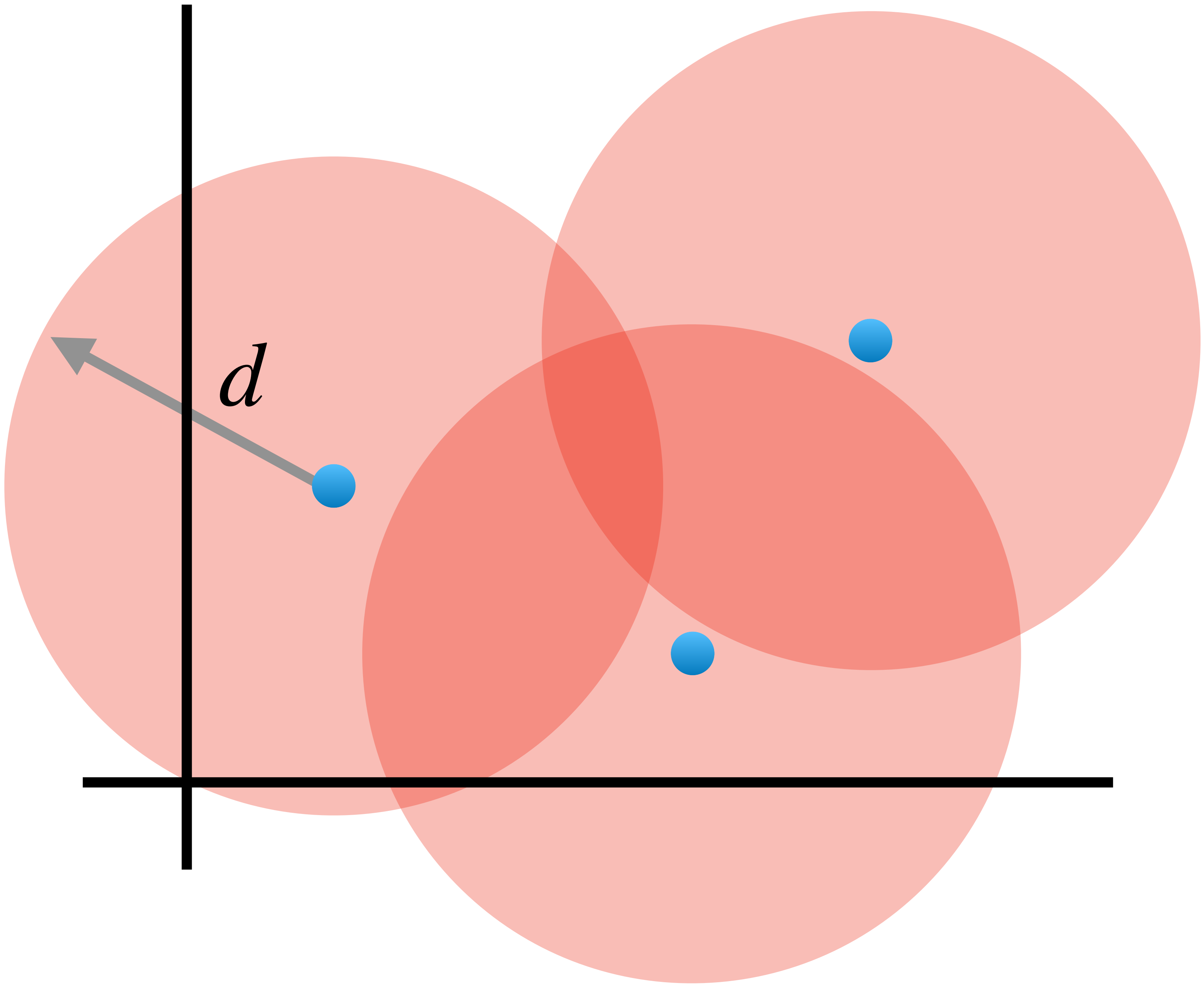

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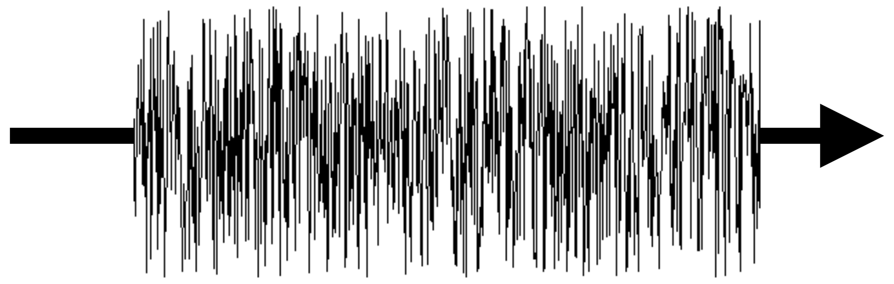
$c \in C$  z

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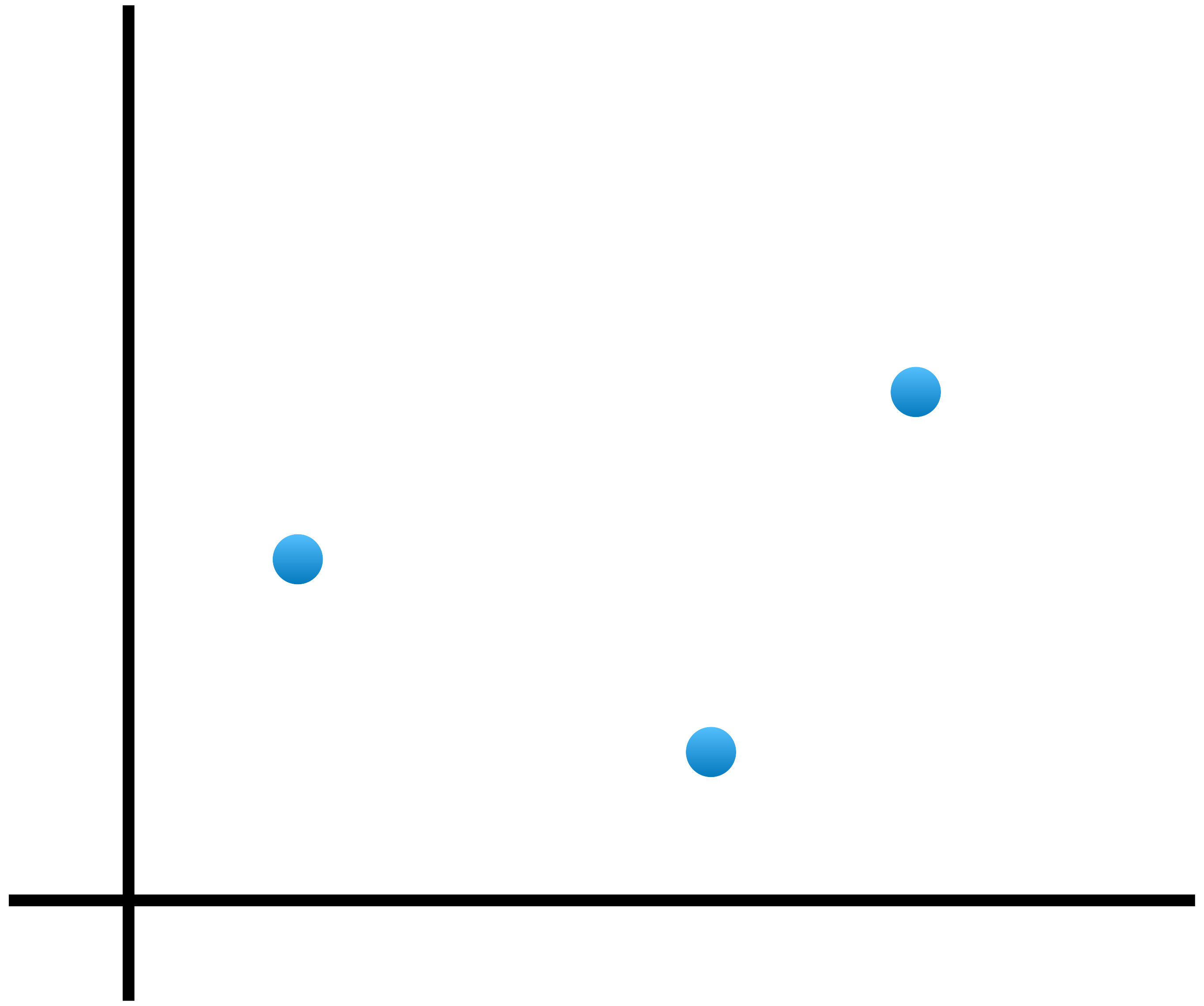

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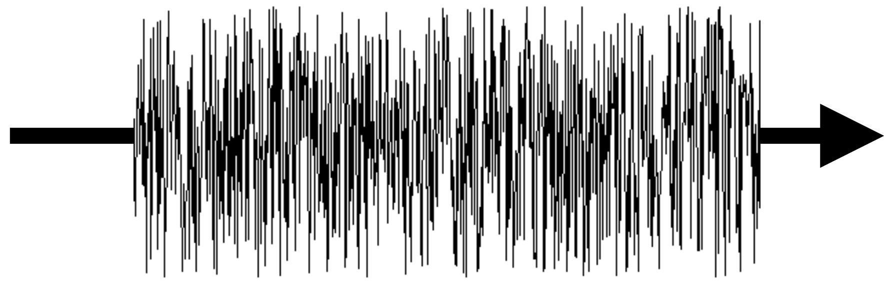
$c \in C$  z

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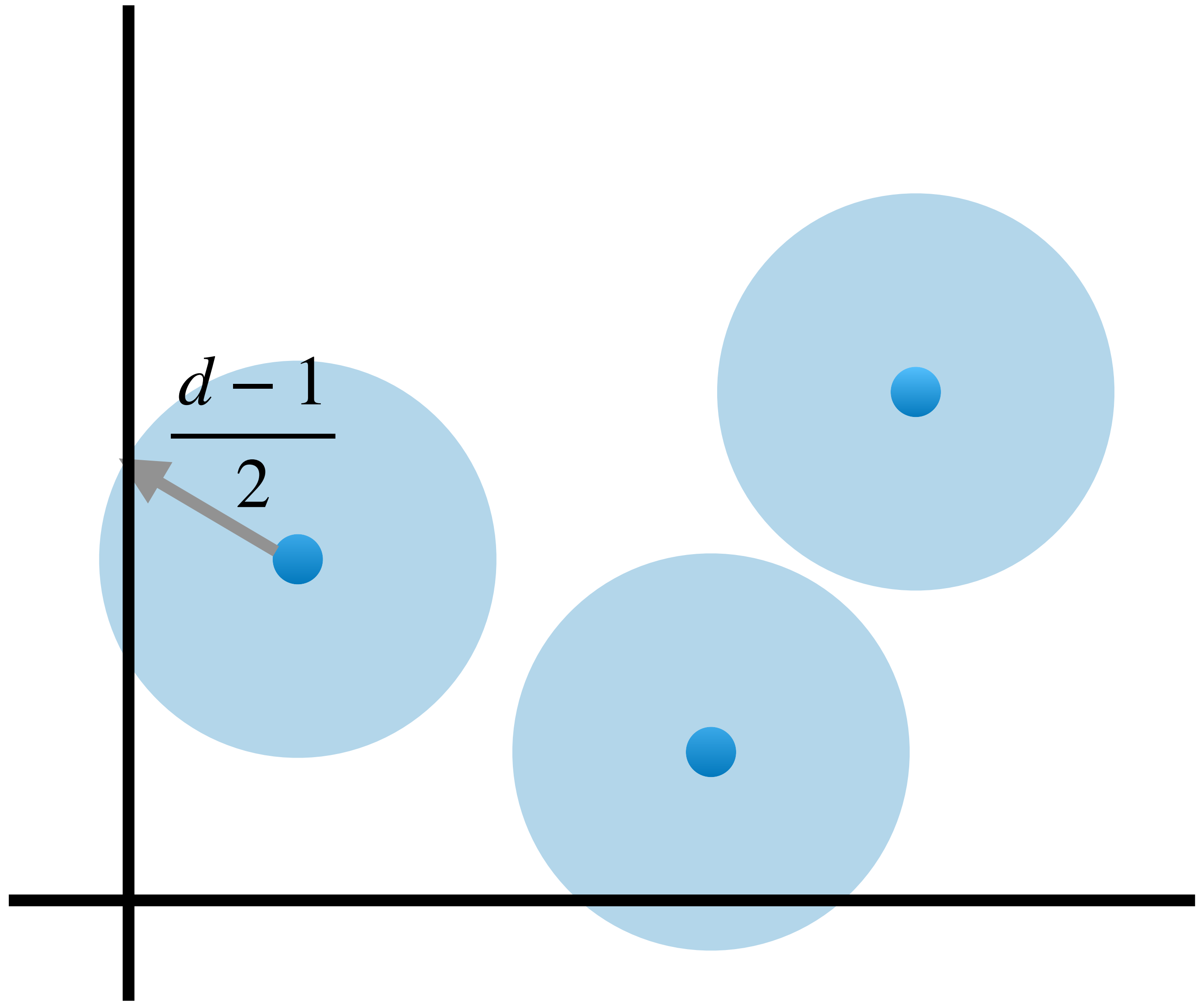

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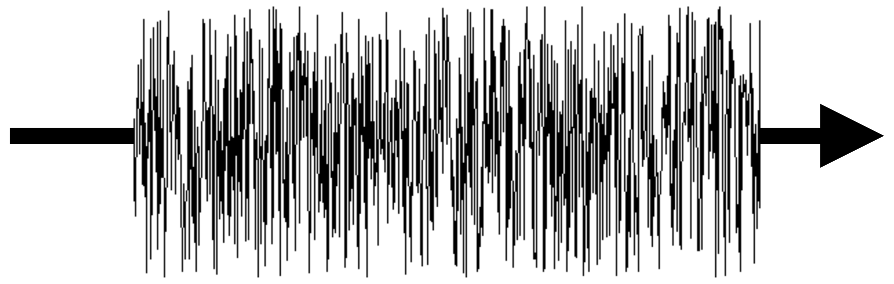
$c \in C$  z

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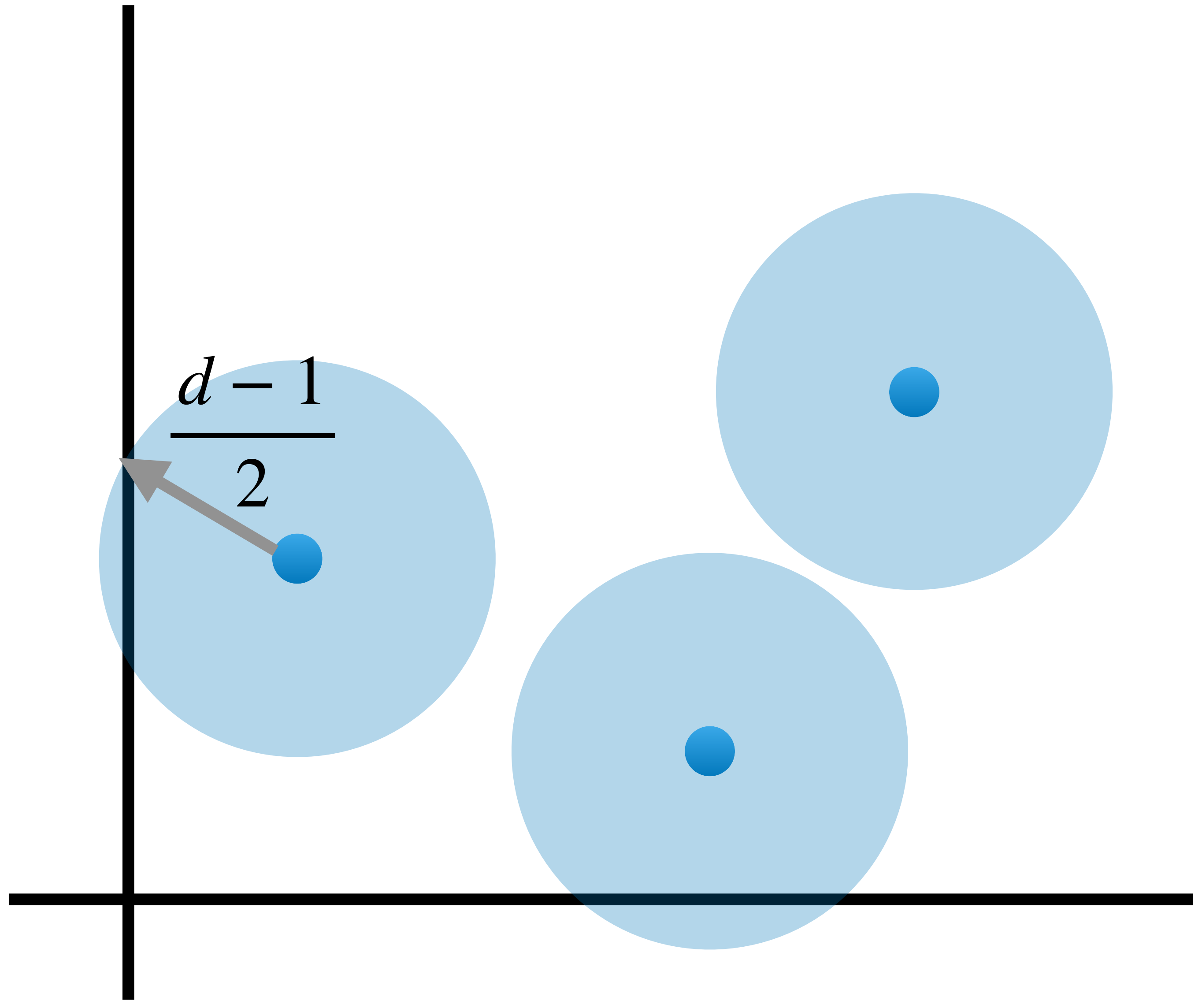

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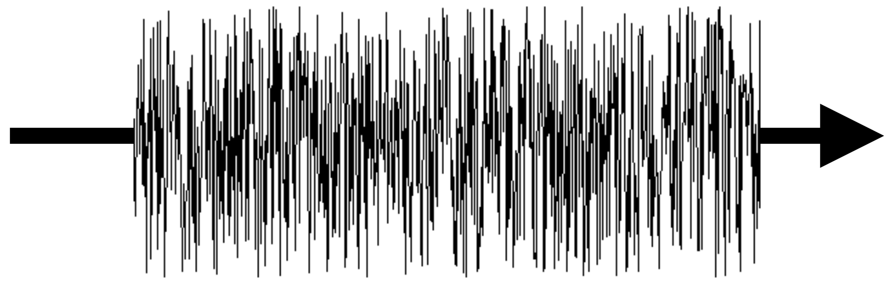
$c \in C$  z

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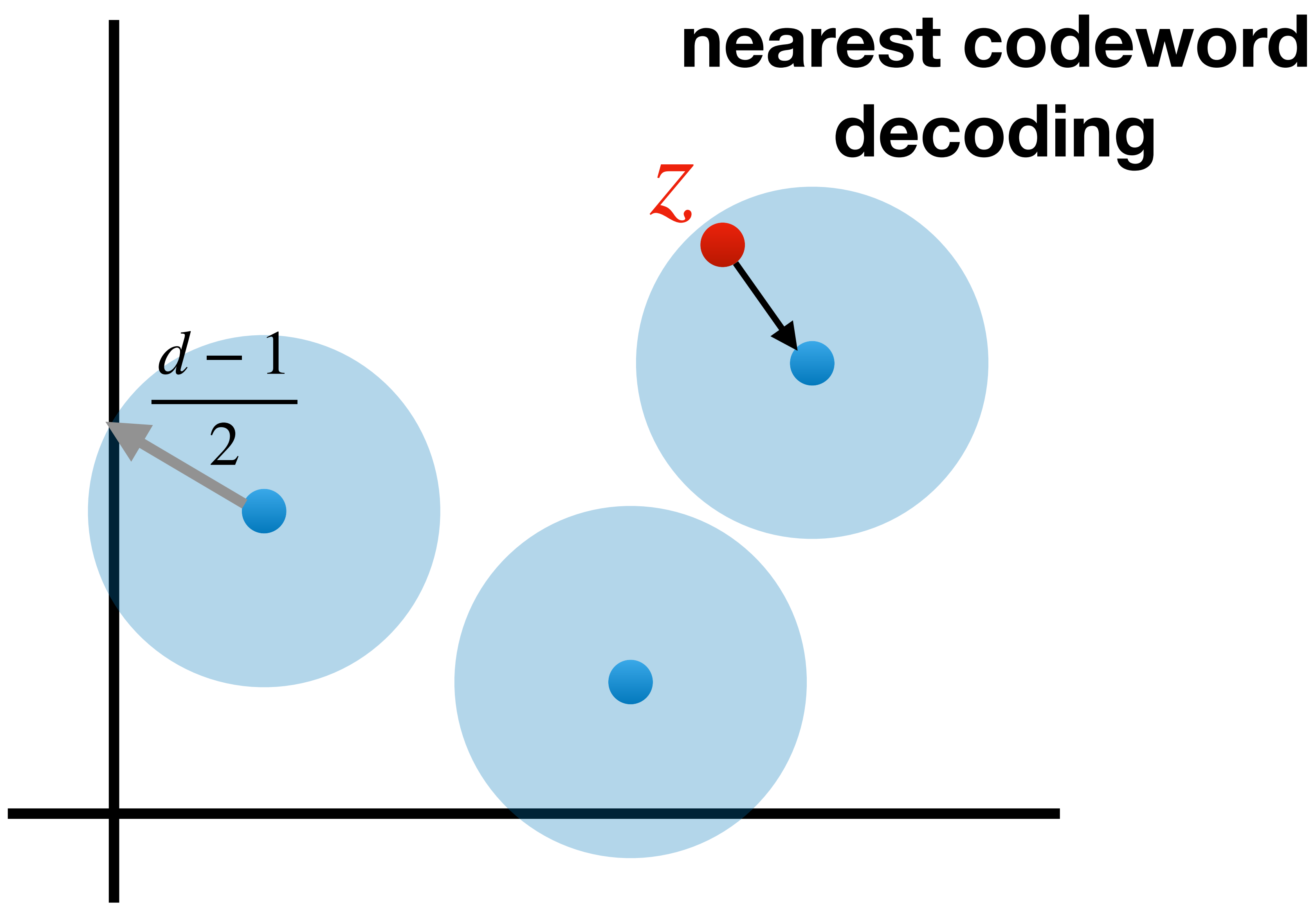

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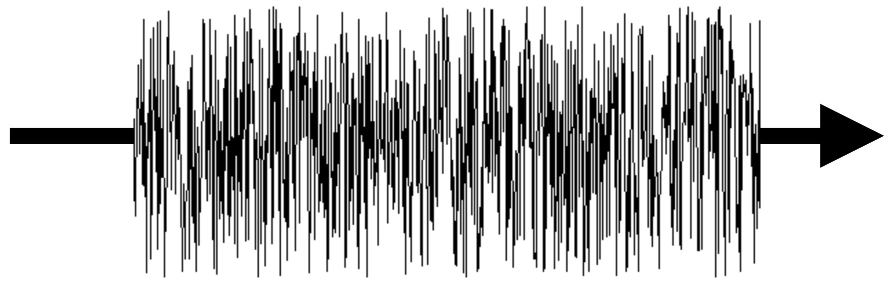
$c \in C$  z

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Fundamental coding problems

Nearest Codeword Problem over $\mathbb{F}_q$ (NCP$_q$)

Input: A generator matrix $G \in \mathbb{F}_q^{n \times k}$, a distance bound $d \geq 0$, and a target vector $t \in \mathbb{F}_q^n$

(YES) There is $c \in C(G)$ such that $\|c - t\|_0 \leq d$

(NO) For every $c \in C(G)$ it holds that $\|c - t\|_0 > d$

Fundamental coding problems

Minimum Distance Problem over $\mathbb{F}_q$ (MDP$_q$)

Input: A generator matrix $G \in \mathbb{F}_q^{n \times k}$ and a distance bound $d \geq 0$

(YES) There is $c \in C(G)$ such that $\|c\|_0 \leq d$ (⟺ C has minimum distance $\leq d$)

(NO) For every $c \in C(G)$ it holds that $\|c\|_0 > d$ (⟺ C has minimum distance $> d$)

MDP$_q$ is NCP$_q$ with target $t = 0$

What is a lattice?

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There exists a **basis** $B \in \mathbb{R}^{n \times k}$ such that $L = \{Bv : v \in \mathbb{Z}^k\}$.

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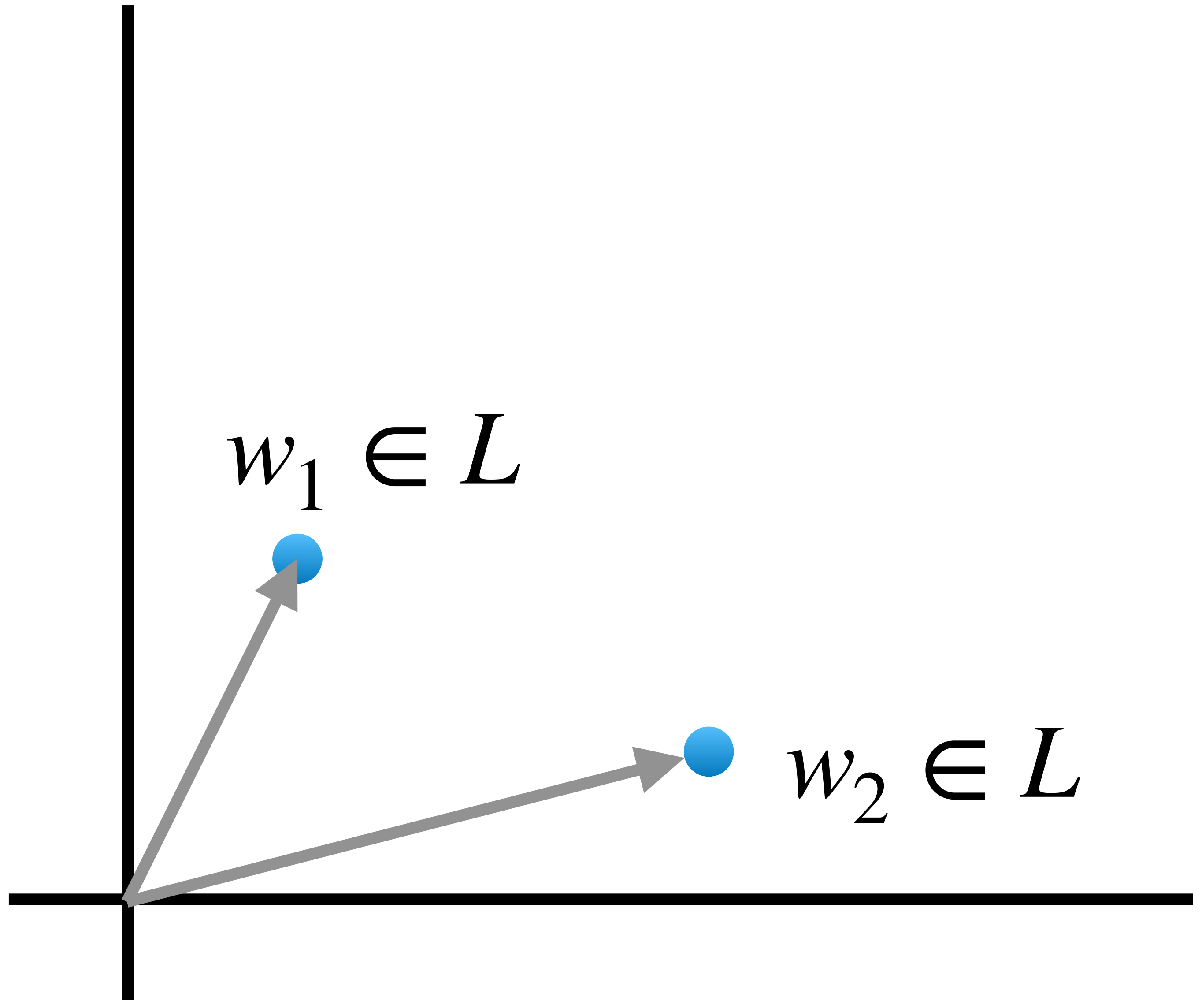

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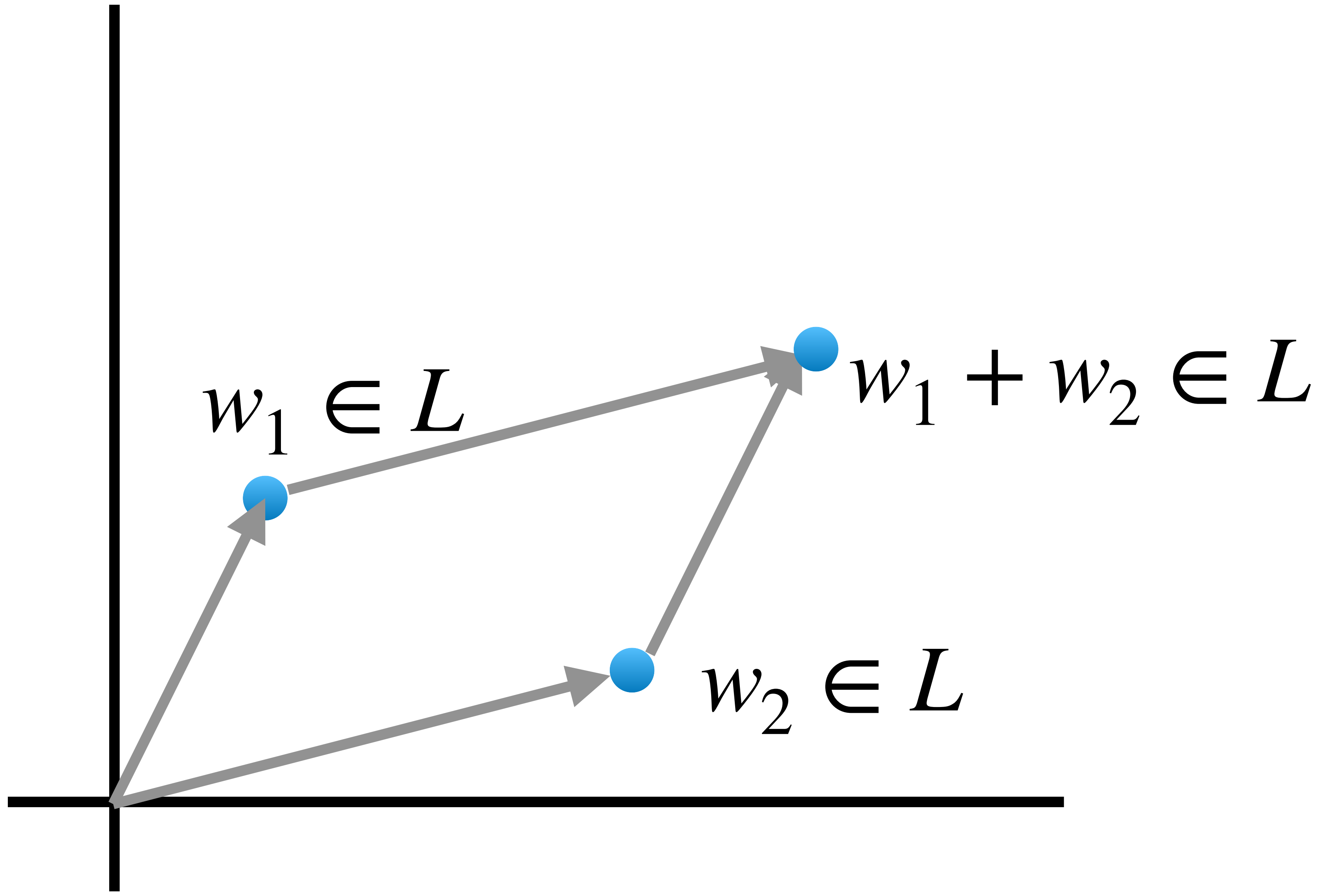

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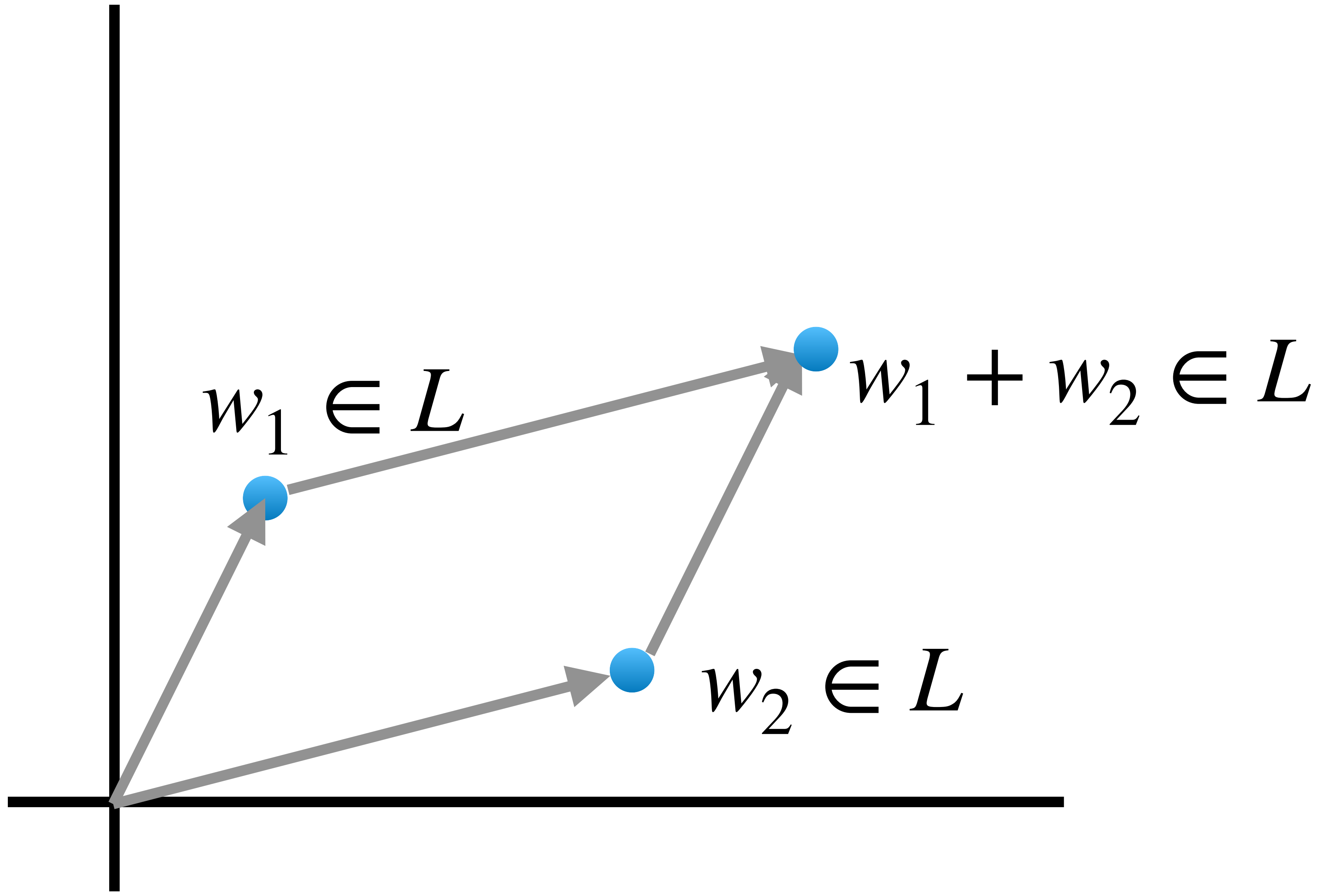

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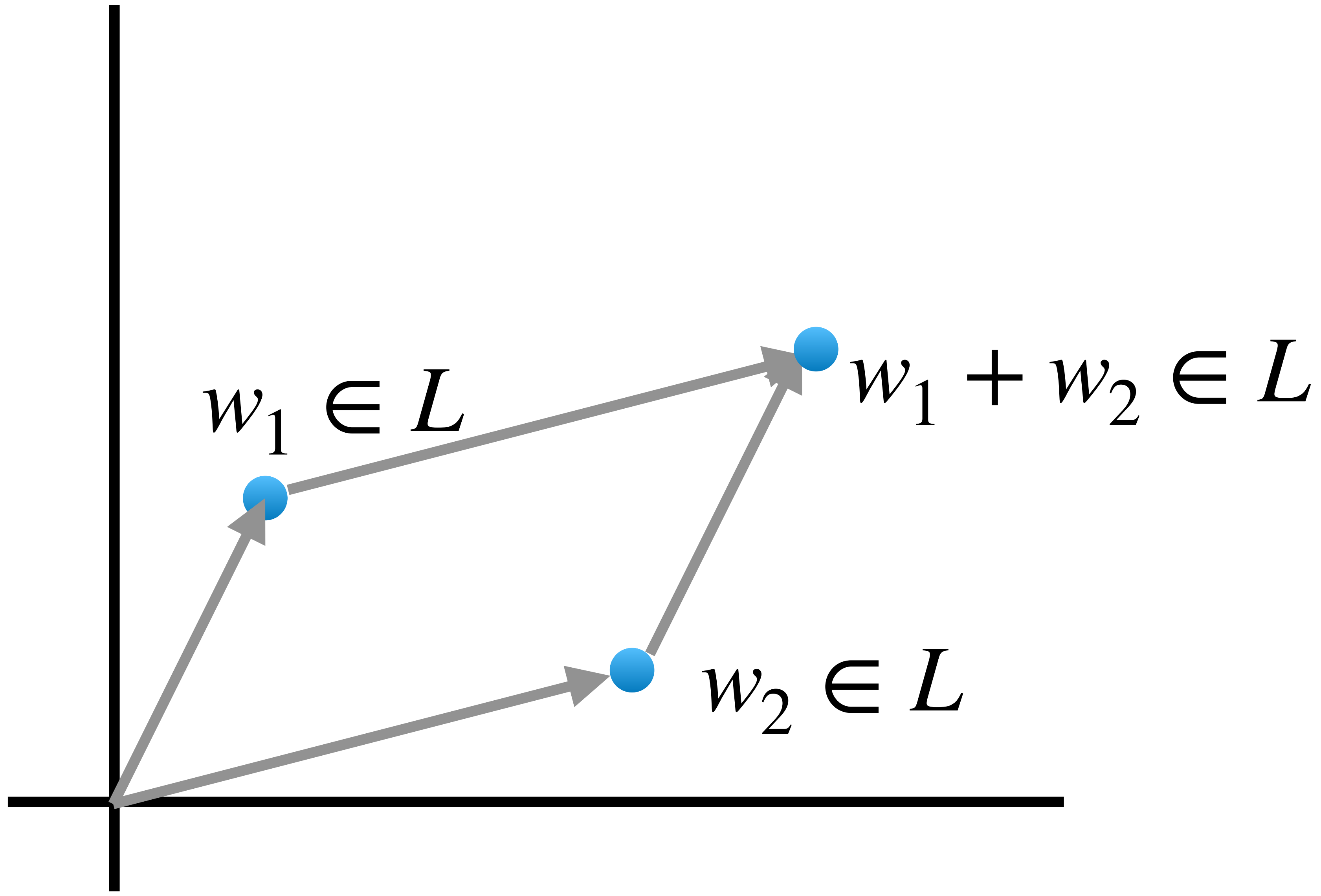

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"Minimum distance" of L:

$$\min_{v \in L \setminus \{0\}} \|v\|_p$$

Fundamental lattice problems

Closest Vector Problem in the ℓ_p norm (CVP$_p$)

Input: A basis $B \in \mathbb{Z}^{n \times k}$, a distance bound $d \geq 0$, and a target vector $t \in \mathbb{Z}^n$

(YES) There is $v \in L(B)$ such that $\|v - t\|_p \leq d$

(NO) For every $v \in L(B)$ it holds that $\|v - t\|_p > d$

Fundamental lattice problems

Shortest Vector Problem in the ℓ_p norm (SVP$_p$)

Input: A basis $B \in \mathbb{Z}^{n \times k}$ and a distance bound $d \geq 0$

(YES) There is $v \in L(B)$ such that $\|v\|_p \leq d$

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SVP$_p$ is CVP$_p$ with target $t = 0$

How hard are all these problems?

Pretty hard!

All **NP-hard**:

- NCP: Berlekamp, McEliece, van Tilborg ’78
- MDP: Vardy ’97
- CVP: van Emde Boas ’81 ($p = 2$)
- SVP: Ajtai ’98 ($p = 2$)

What if we only want approximate solutions?

For example, γ-approximate SVP$_p$ for approximation factor $\gamma \geq 1$:

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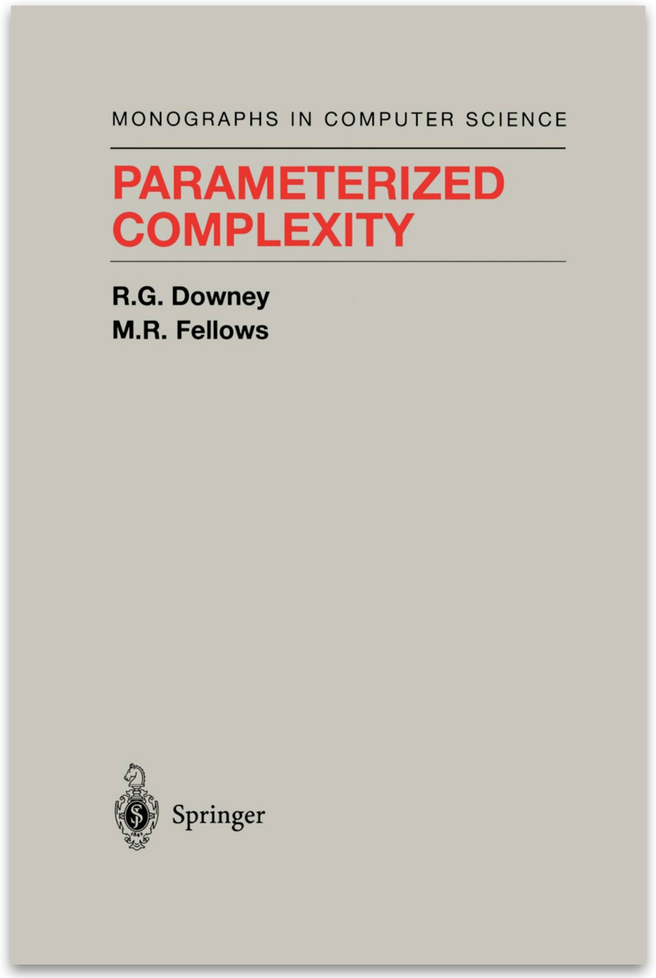
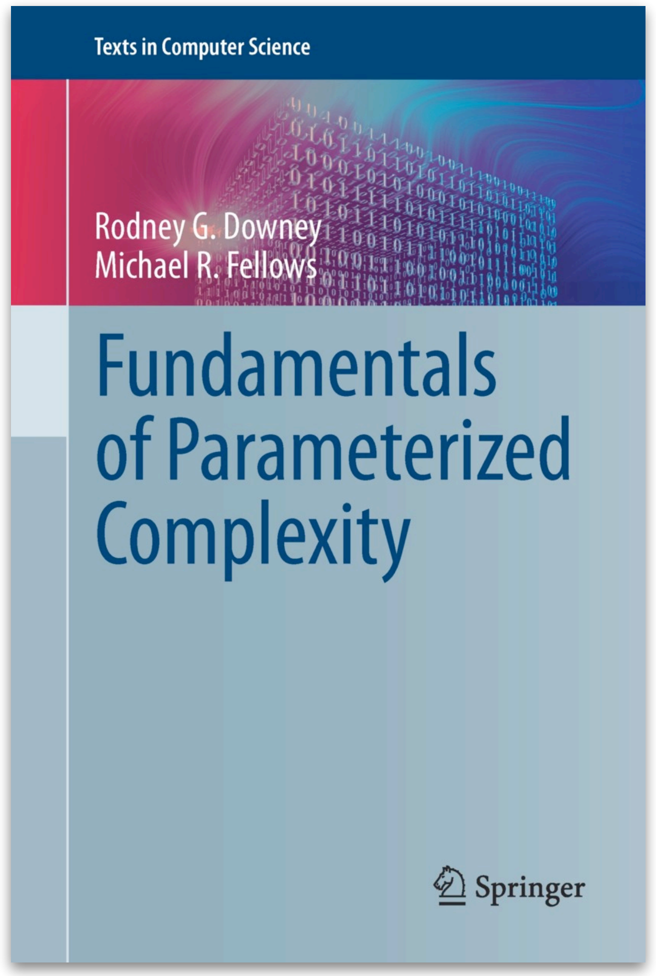
Still pretty hard!

All **NP-hard** for arbitrary constant approximation factor γ (and beyond):

- NCP: Håstad ’01
- MDP: Dumer, Micciancio, Sudan ’03
- CVP/SVP: Micciancio ’00; Khot ’05; Haviv, Regev ’12 ($p \geq 1$)

Parameterized complexity





Problem Π is NP-hard. Does this mean that “real-world” instances of Π are computationally intractable? **Not necessarily…**

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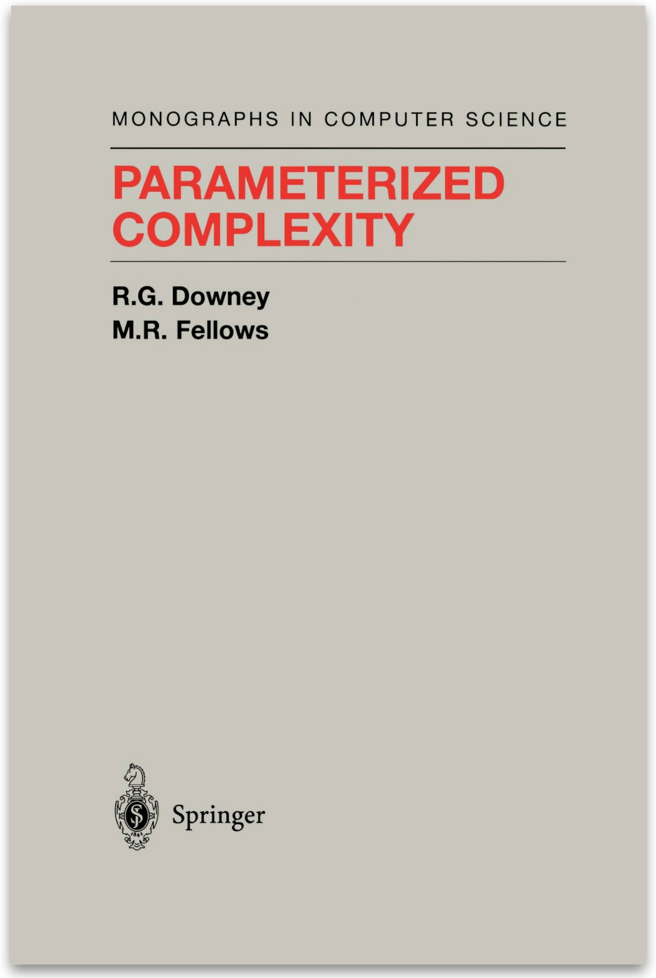

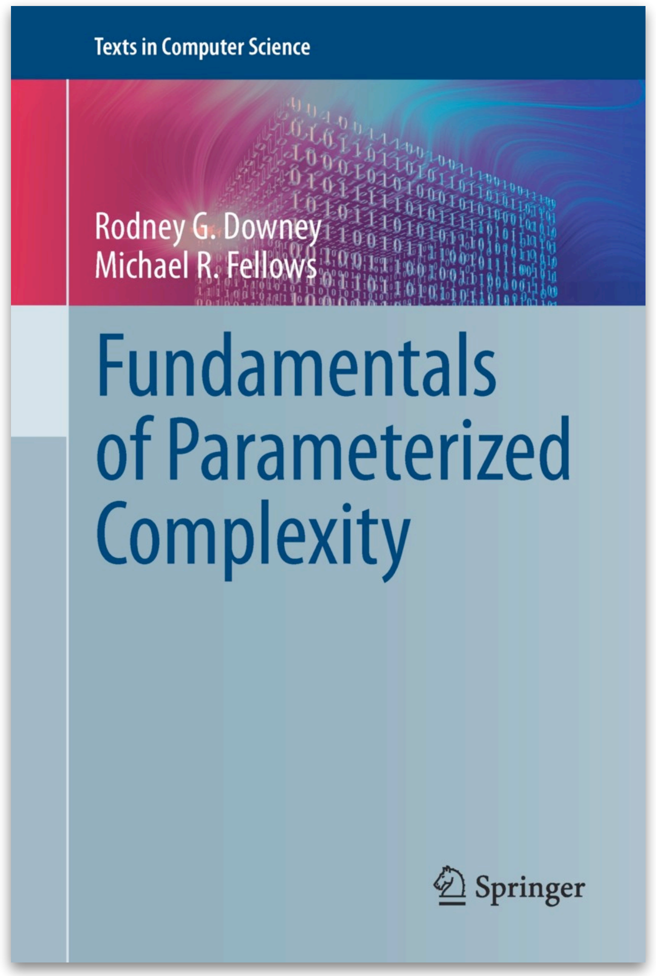

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YES if G has vertex cover of size $\leq k$, **NO** otherwise.

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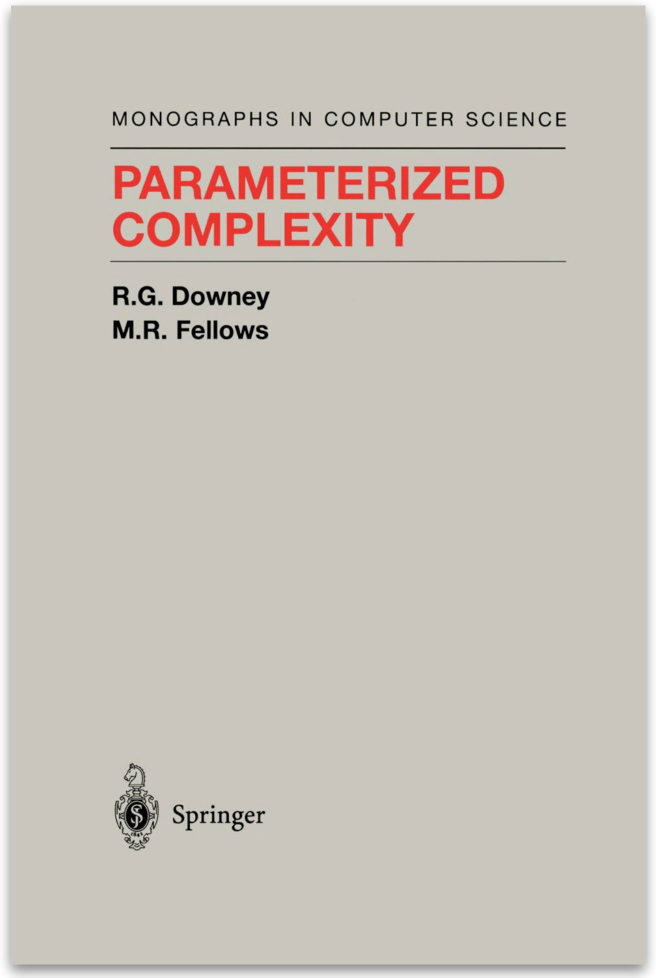

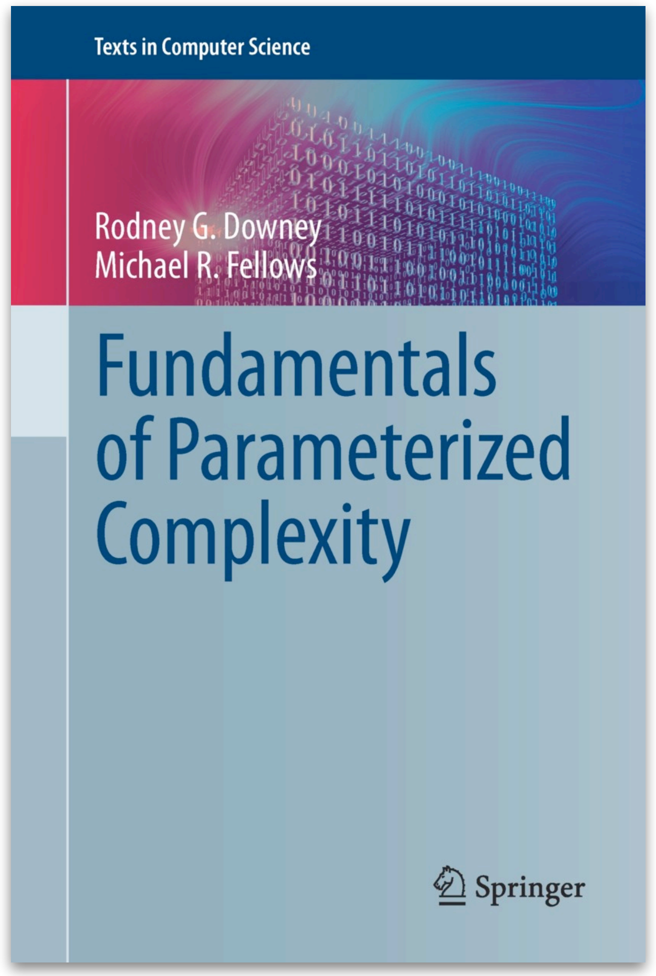

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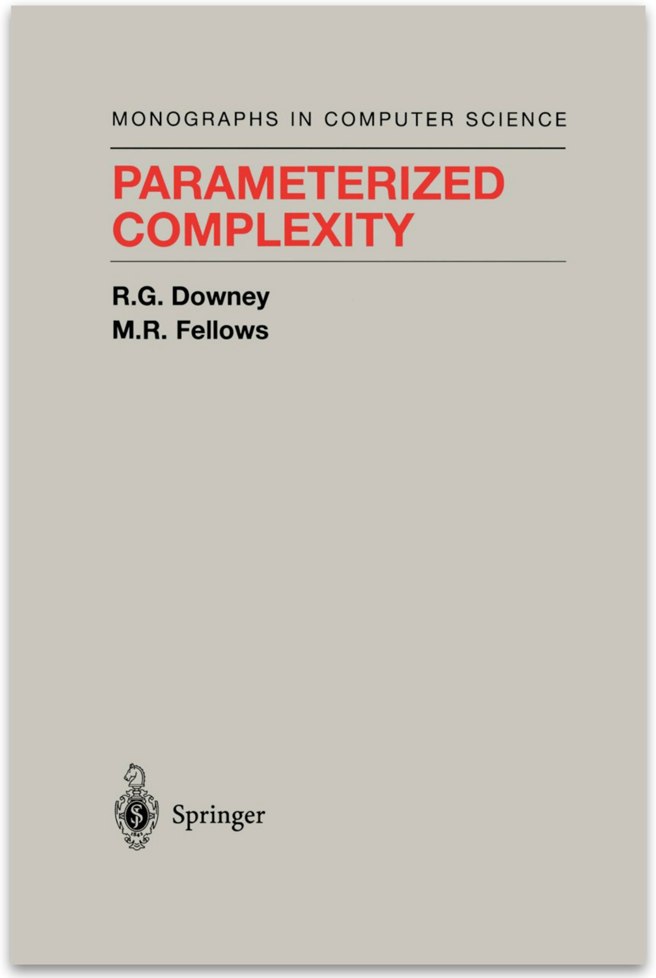

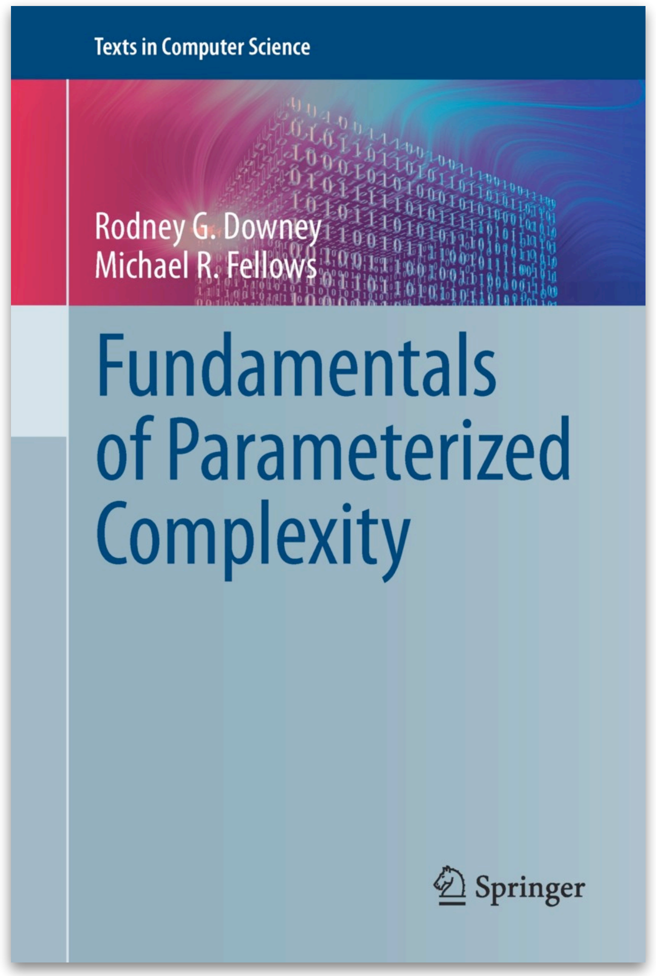

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$\Longrightarrow$ **Practical algo for Vertex Cover instances with small k!**

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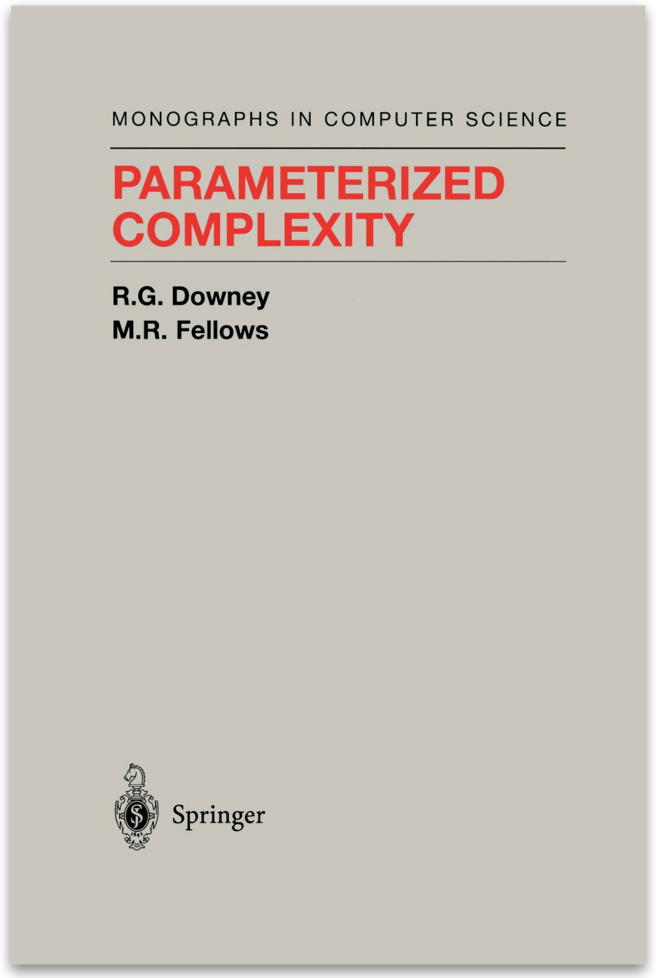
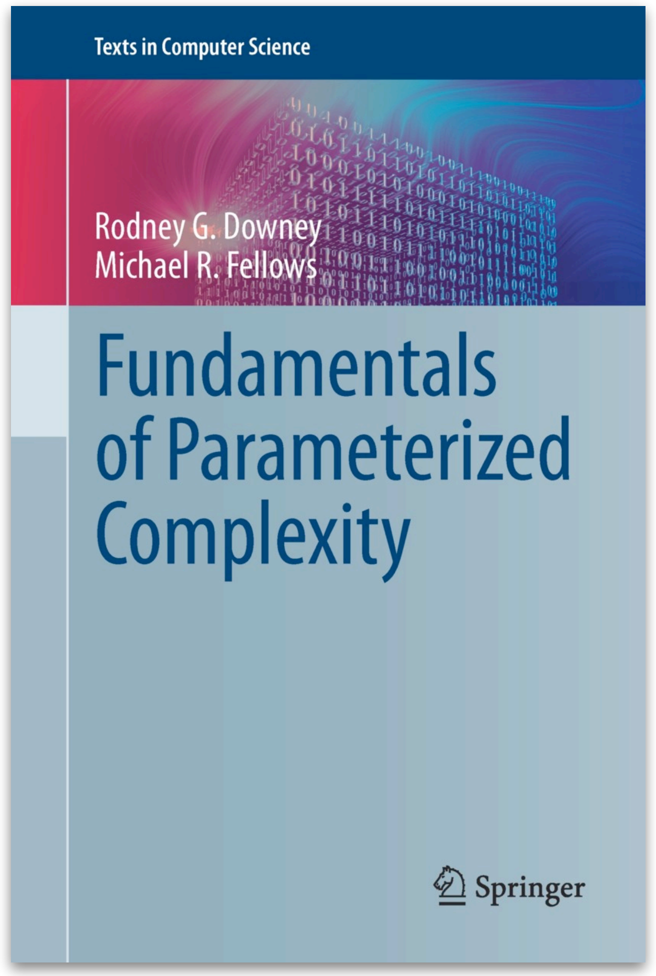




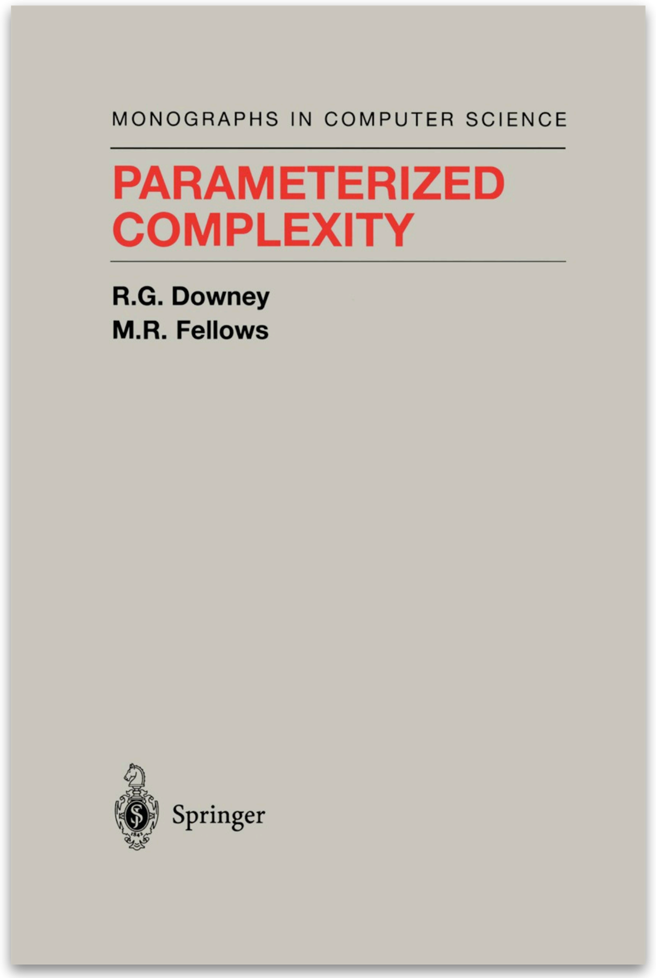
See input to problem Π as (x, k), where k is the **parameter of interest.**

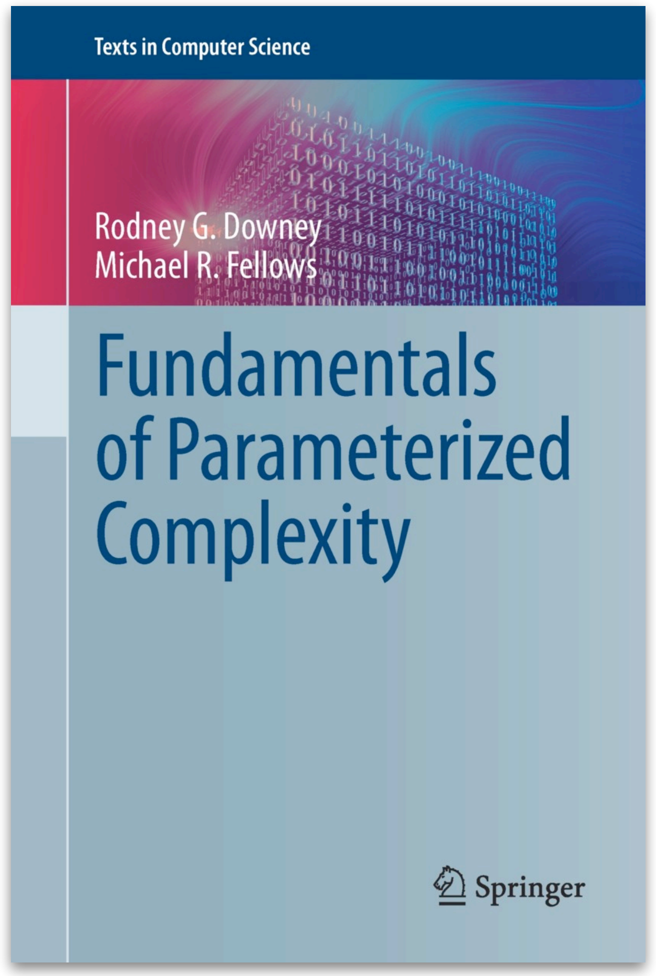
A problem Π parameterized by k is Fixed-Parameter Tractable (FPT) if there is an associated algorithm running in time

$$f(k) \cdot |x|^{O(1)}$$

for **some** function f.

Parameterized complexity





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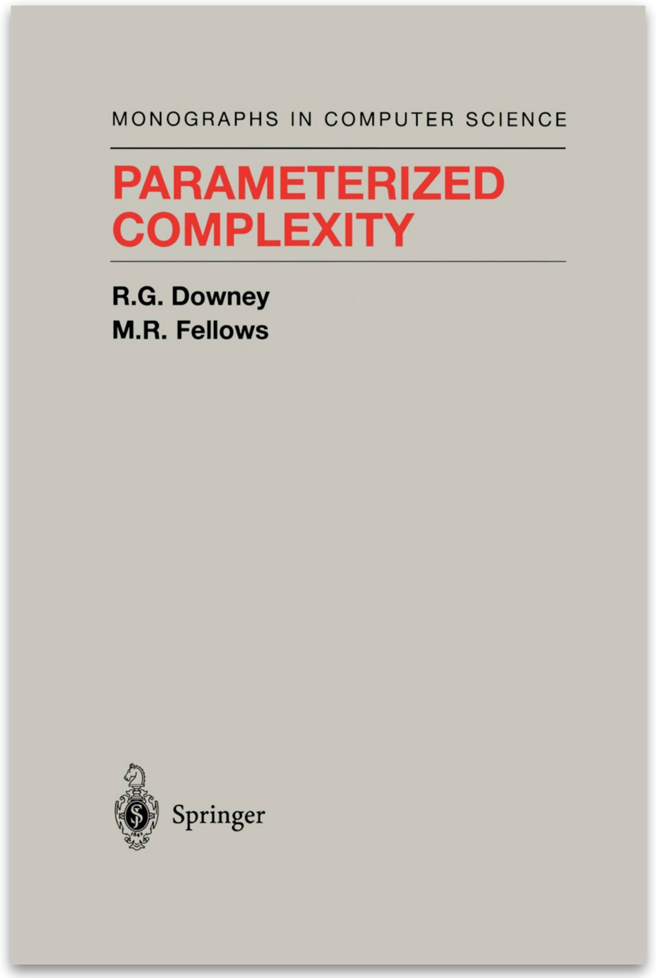

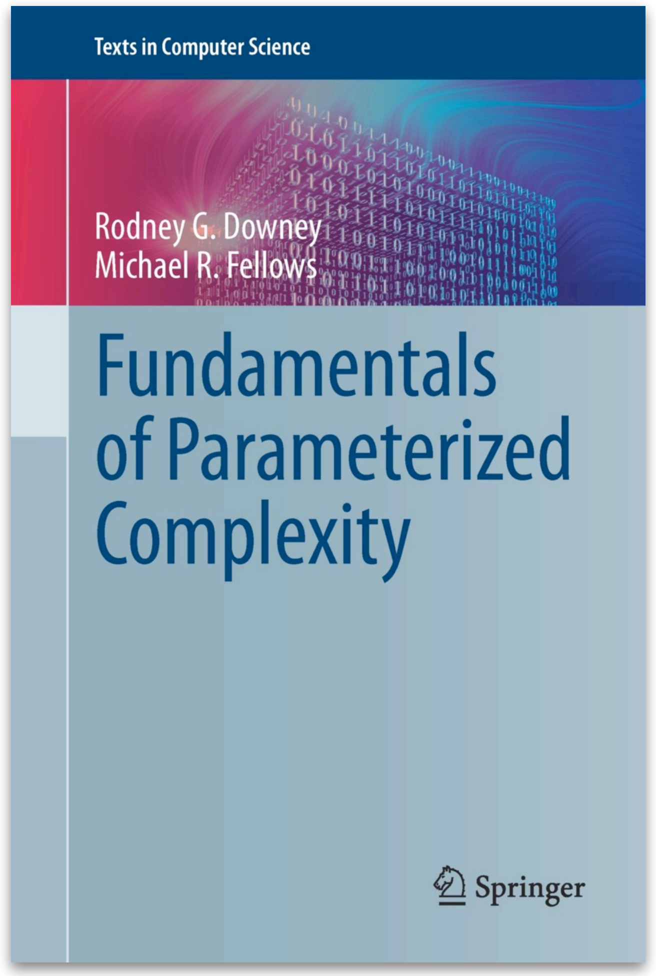

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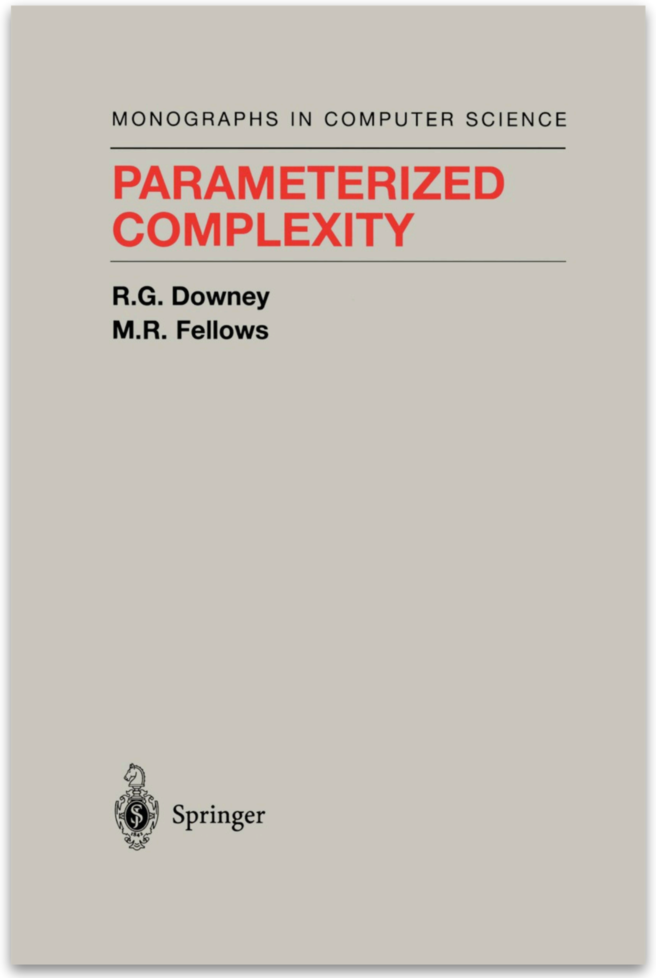

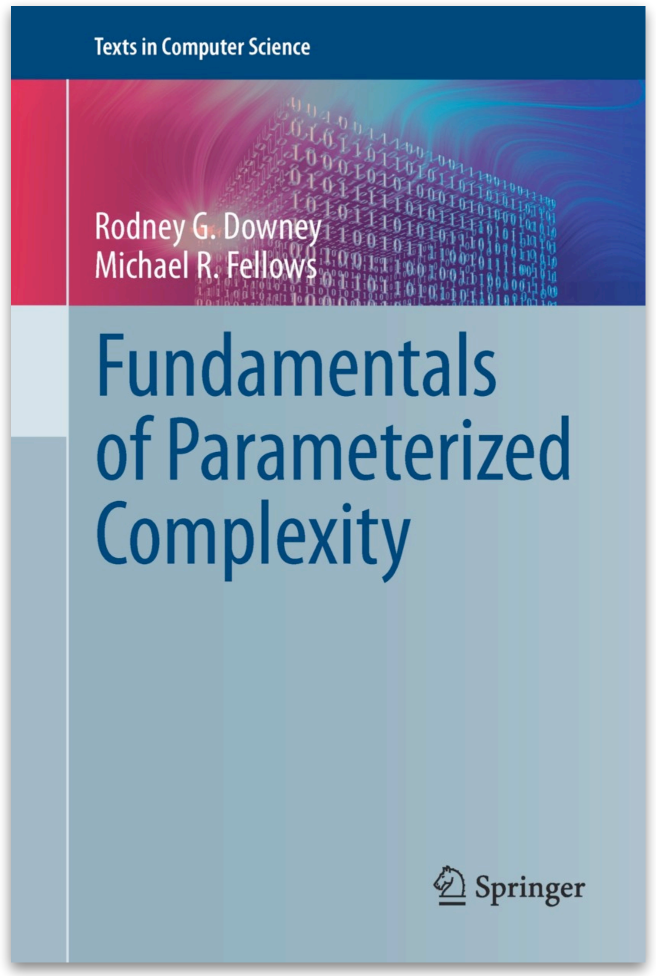

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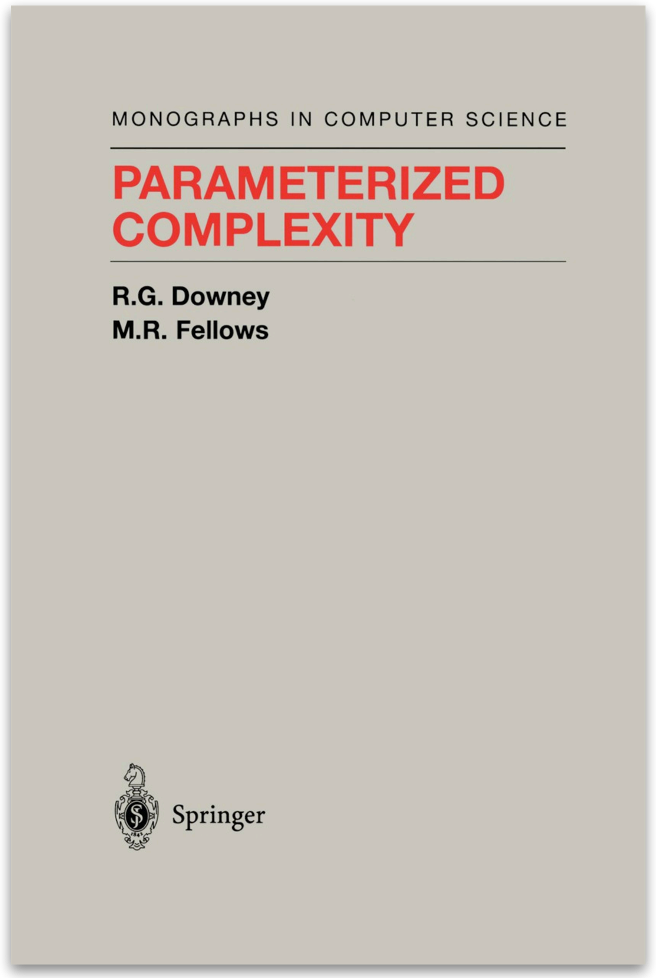
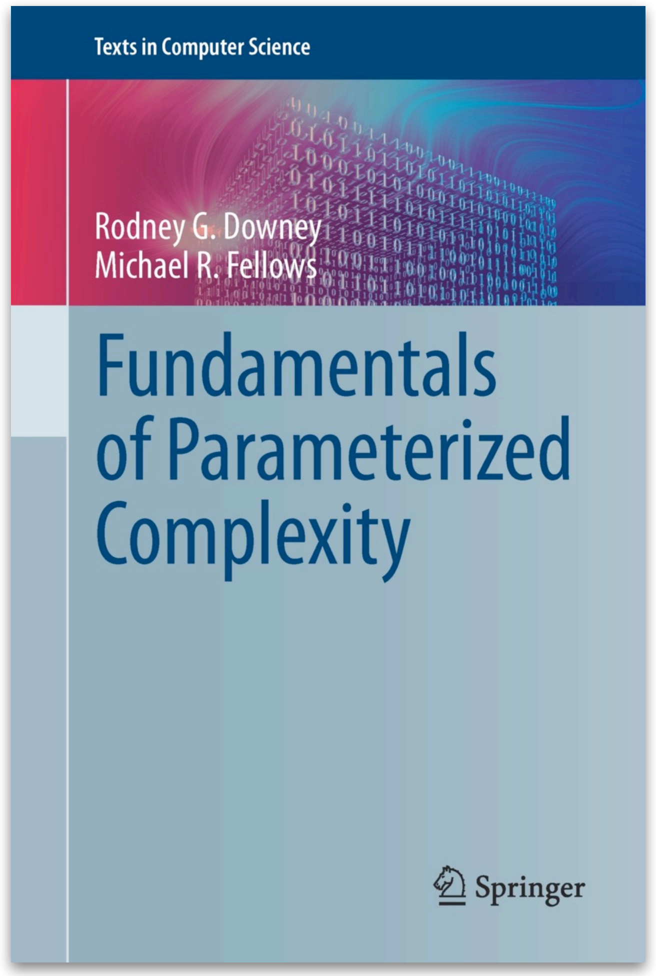
Are there interesting problems believed not to be FPT?

Parameterized complexity





Clique

Input: n-vertex graph G and **parameter of interest** k

YES if G has clique of size k, **NO** otherwise.

Clique is the **canonical hard** parameterized problem. A parameterized problem Π is "hard" if there is an "FPT reduction" from Clique to Π.

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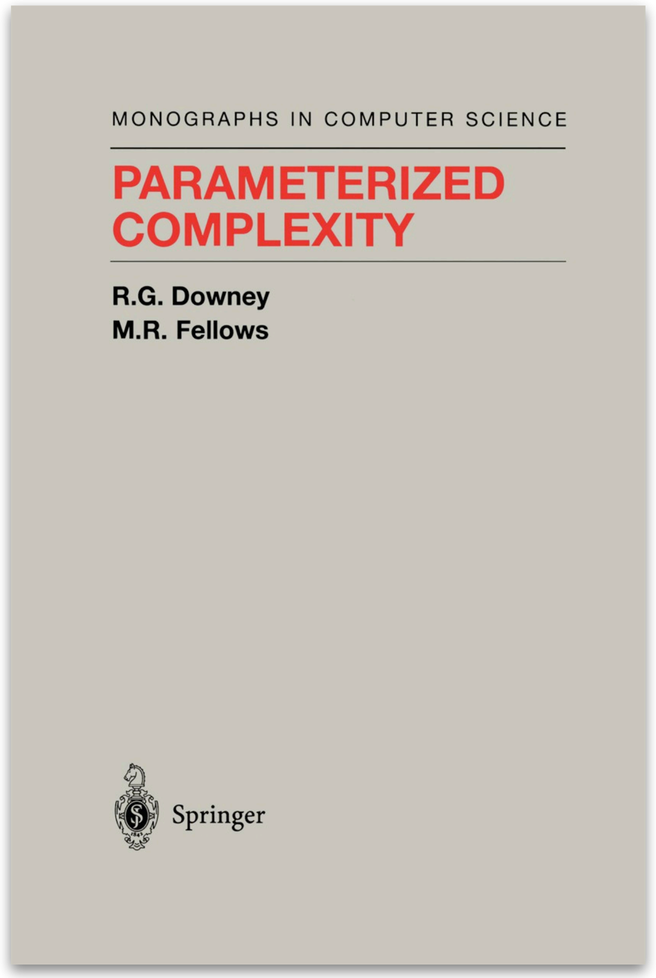

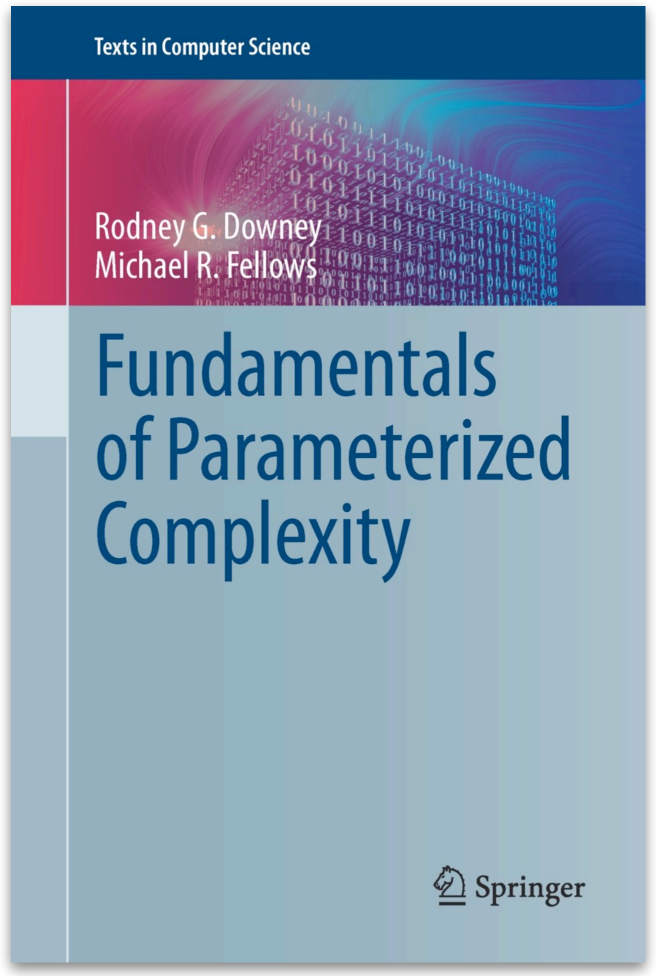

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FPT reduction:

(G, k) Clique instance $\xrightarrow{\text{algorithm running in time } f(k) \cdot n^c}$ (x, k') Π instance

YES/**NO** instances mapped to **YES**/**NO** instances & $k' = g(k)$

Dictionary

Unparameterized world

- The class P
- The class NP
- Π is NP-hard if there is a polynomial-time reduction from 3SAT to Π

Parameterized world

- The class FPT
- The class $W[1]$
- Π is $W[1]$-hard if there is an **FPT-reduction** from Clique to Π

FPT reduction:

(G, k)
Clique instance

$\xrightarrow{\text{algorithm running in time } f(k) \cdot n^c}$

(x, k')
Π instance

YES/NO instances mapped to YES/NO instances & $k' = g(k)$

Parameterized hardness of coding and lattice problems

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Codes:

γ-NCP$_q$ is $W[1]$-hard for all q and γ

γ-MDP$_2$ is $W[1]$-hard for all γ

Lattices:

γ-CVP$_p$ is $W[1]$-hard for all $p \geq 1$ and γ

γ-SVP$_p$ is $W[1]$-hard for all $p > 1$ and some (small) $\gamma(p) > 1$

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what about γ-MDP_q for $q > 2$?

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what about γ-SVP_p for large γ?

and SVP_1?

Our results

Codes:

γ-MDP$_q$ is $W[1]$-hard for **all q and** all γ

Lattices:

γ-SVP$_1$ is $W[1]$**-hard for any** $\gamma < 2$

γ-SVP$_p$ is $W[1]$-hard for $p > 1$ and **all** $\gamma > 1$

NP-hardness of approximating SVP

A pretty cool approach of Khot

Let's take for granted that γ-CVP$_p$ is NP-hard, and reduce it to γ'-SVP$_p$.

Want: Transform a γ-CVP$_p$ instance (B, d, t) into a γ'-SVP$_p$ instance (B', d').

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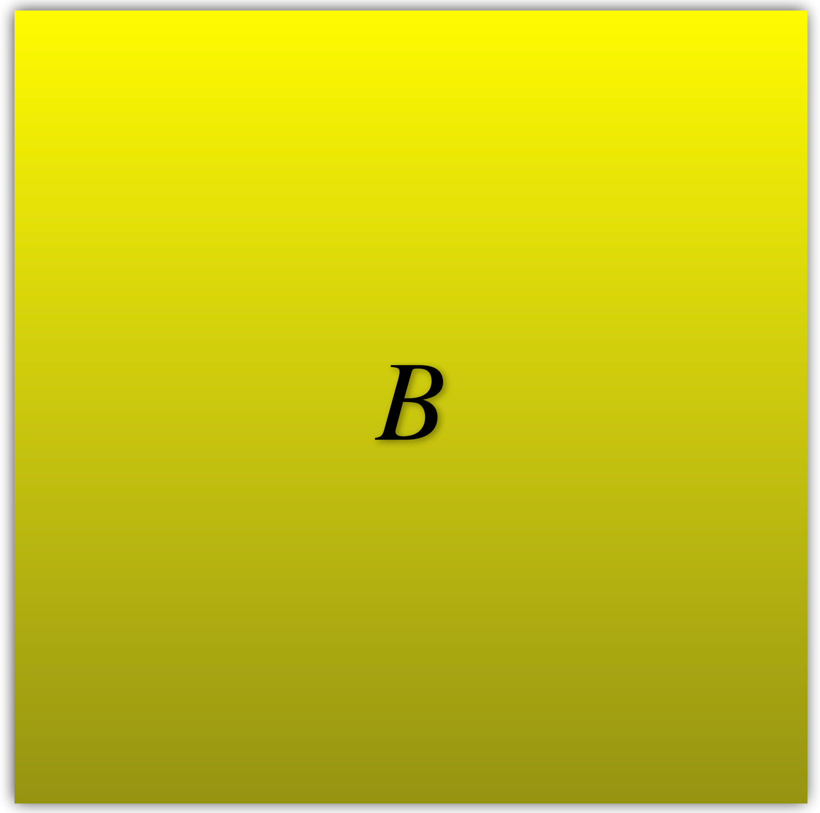

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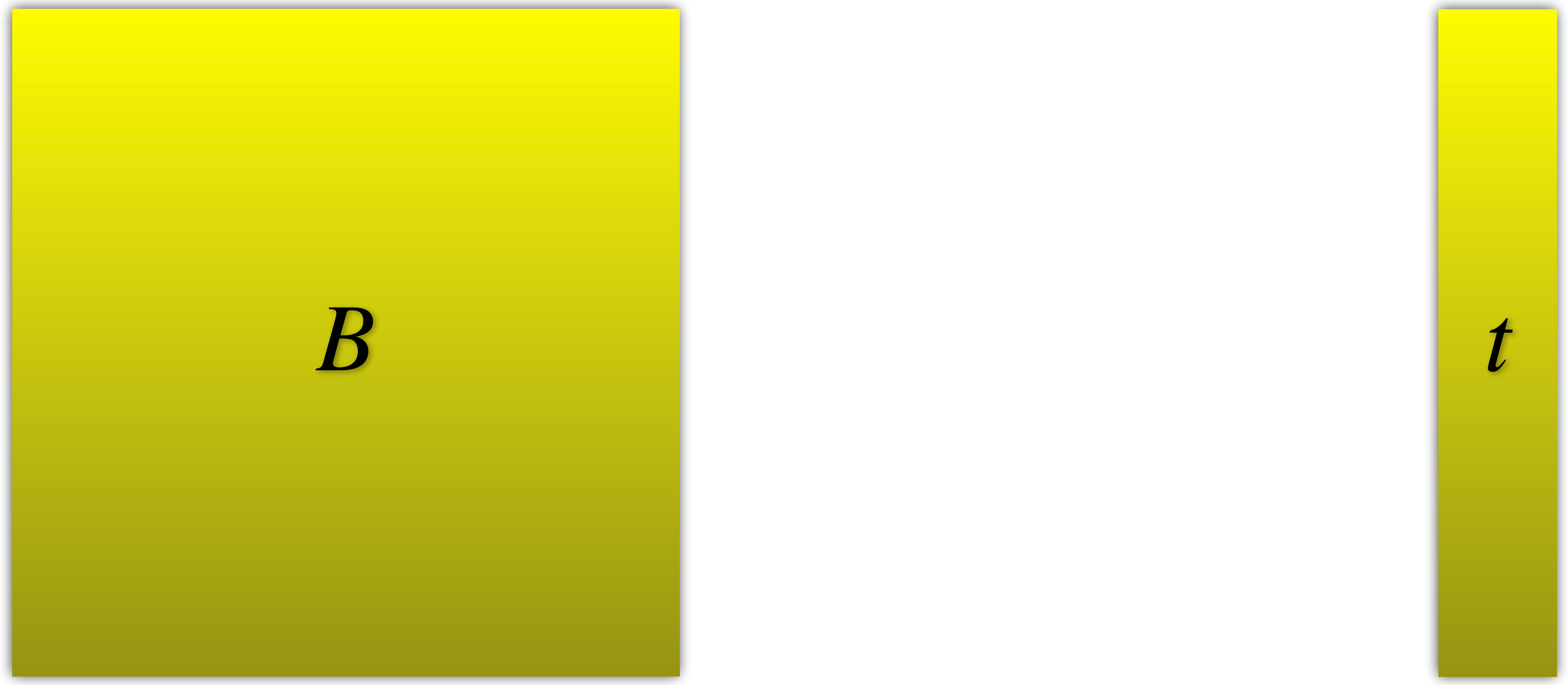

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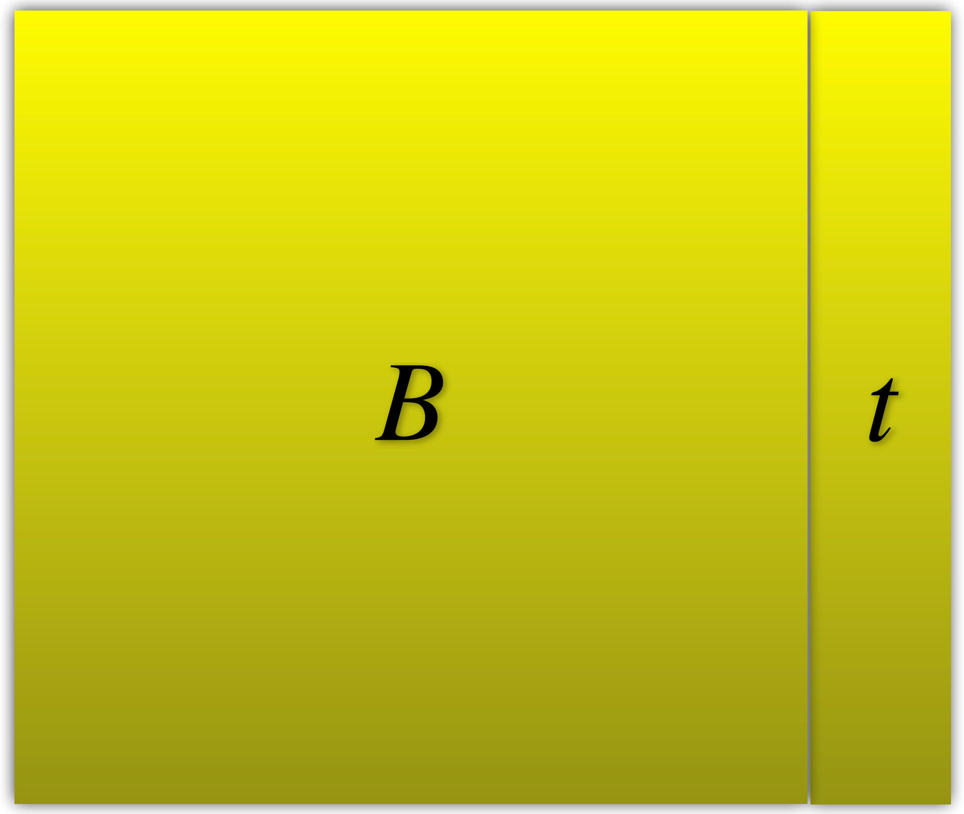

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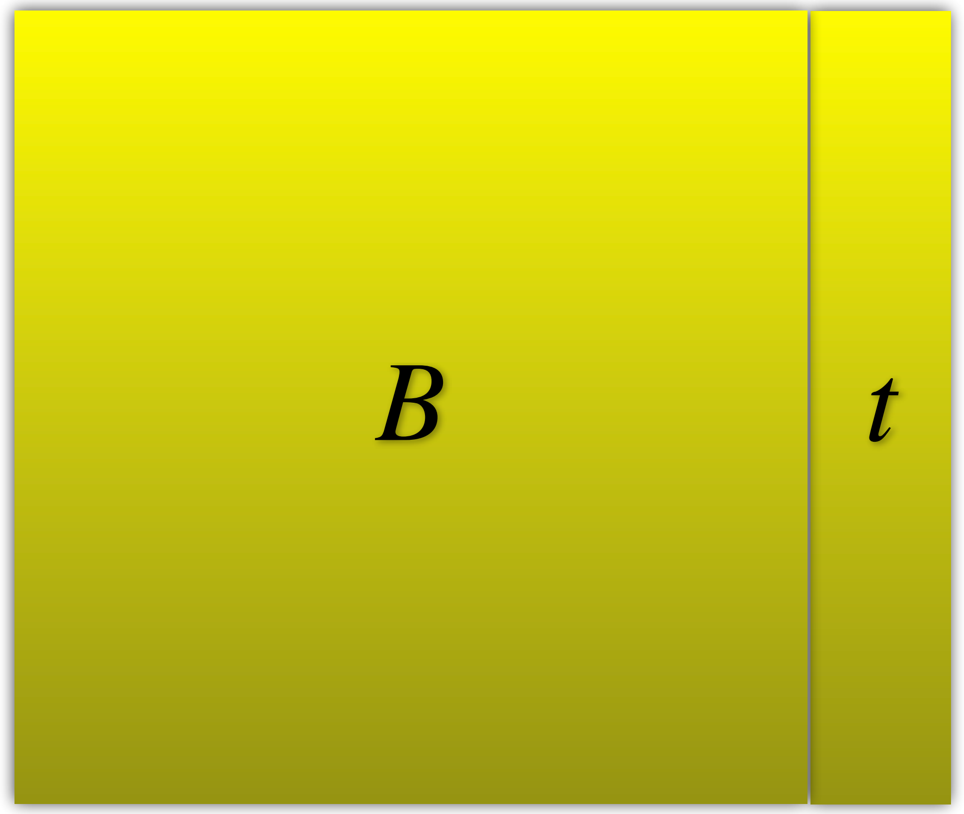

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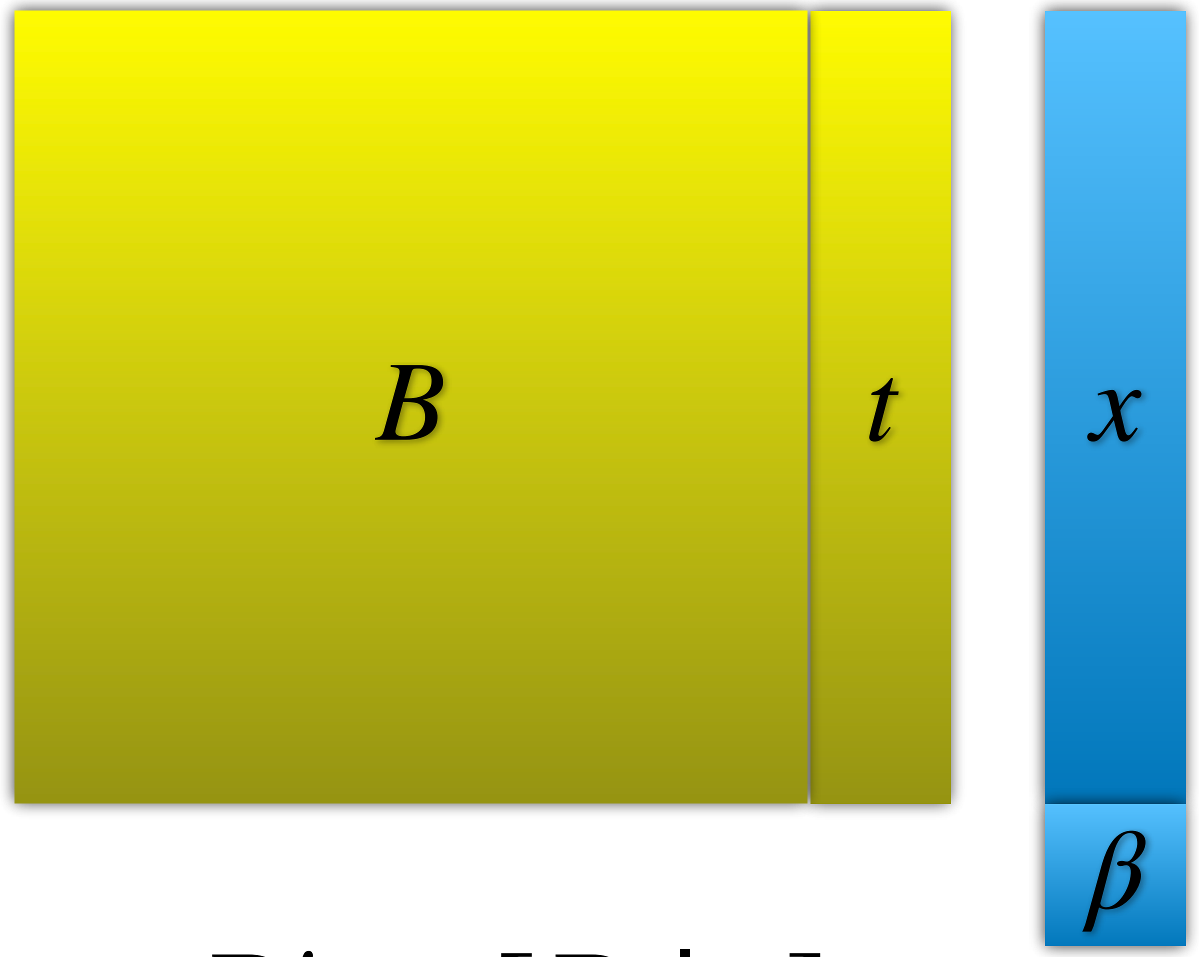

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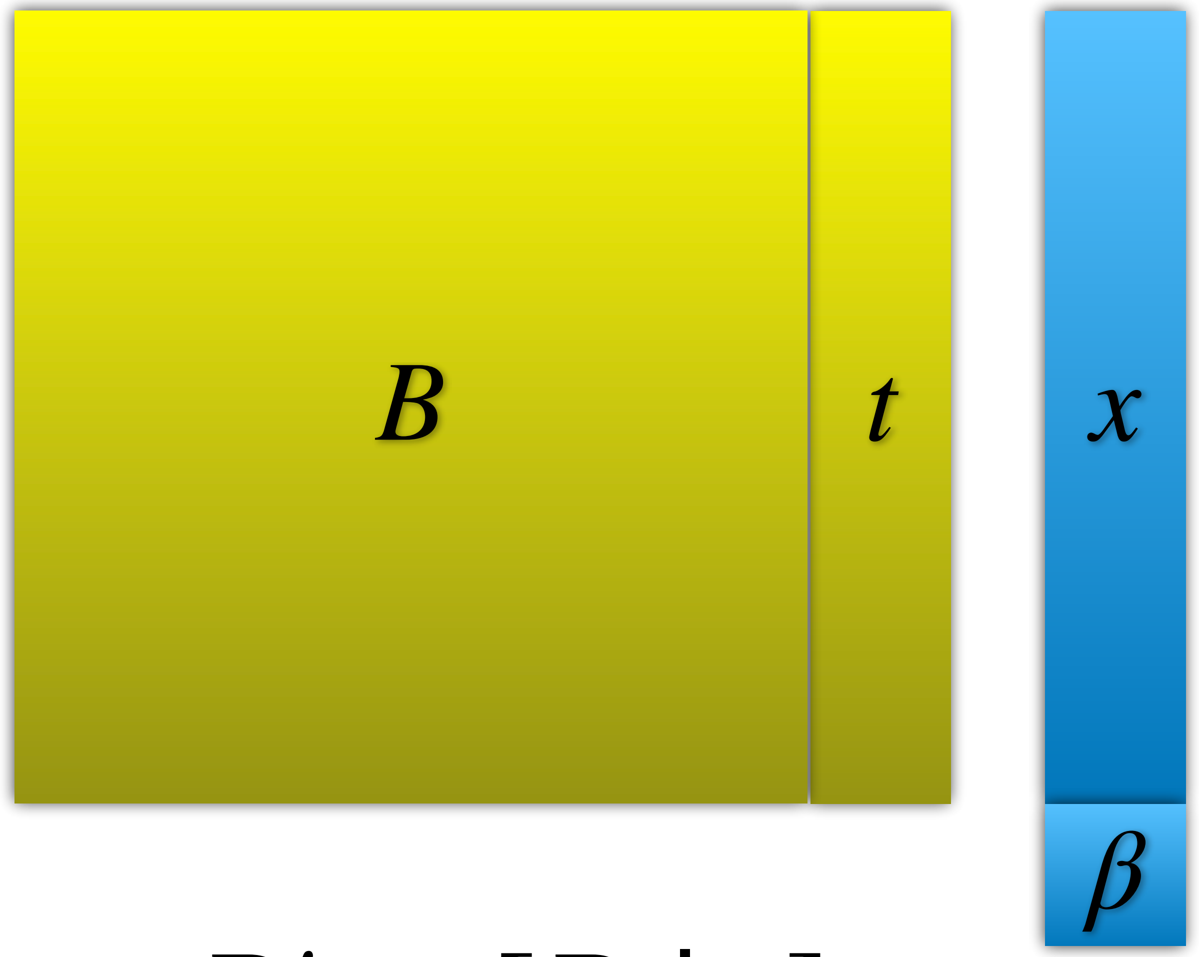

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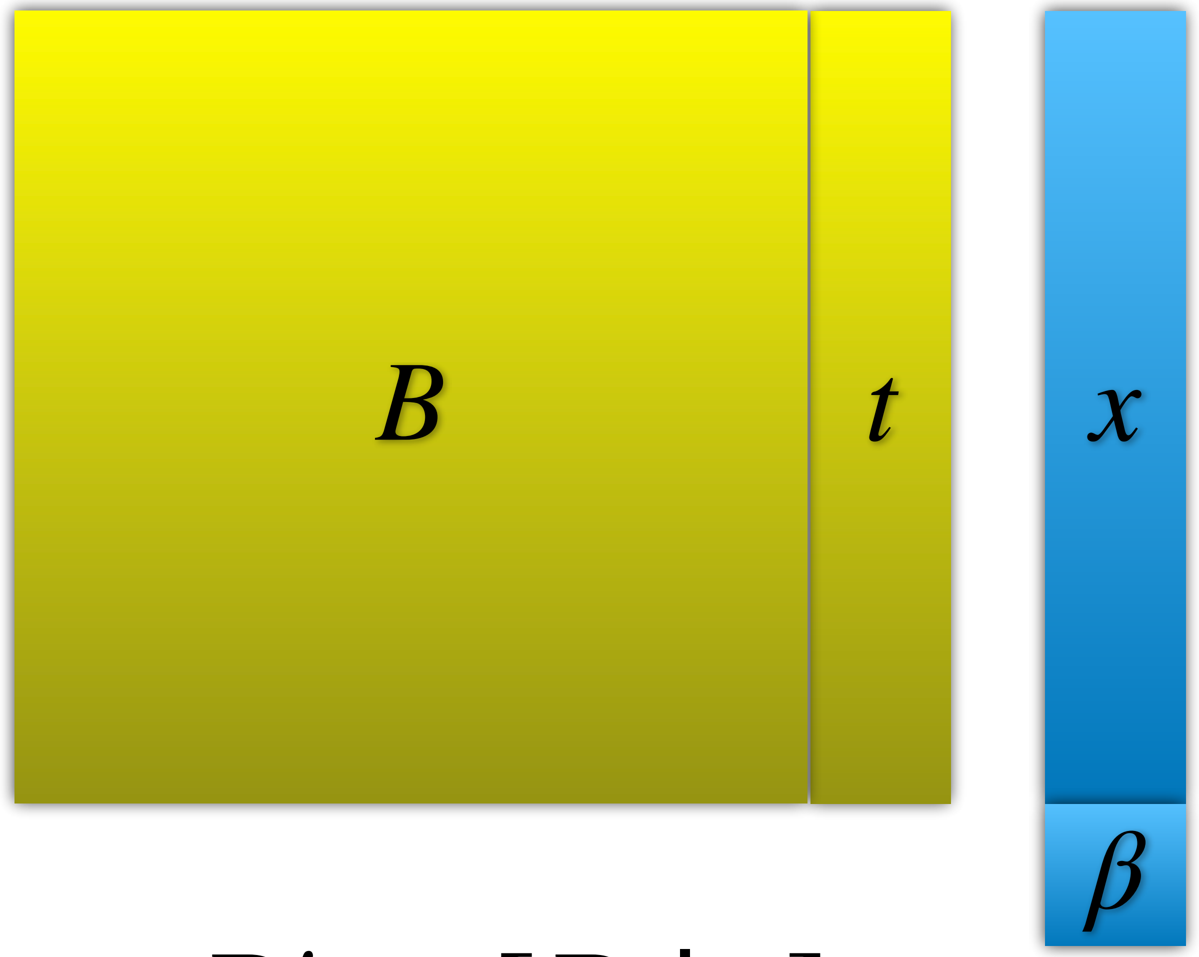

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If (B, d, t) is NO instance of γ-CVP$_p$ …

- $\beta \neq 0$: Then $||Bx + \beta t||_p$ is large
- $\beta = 0$: No guarantees…

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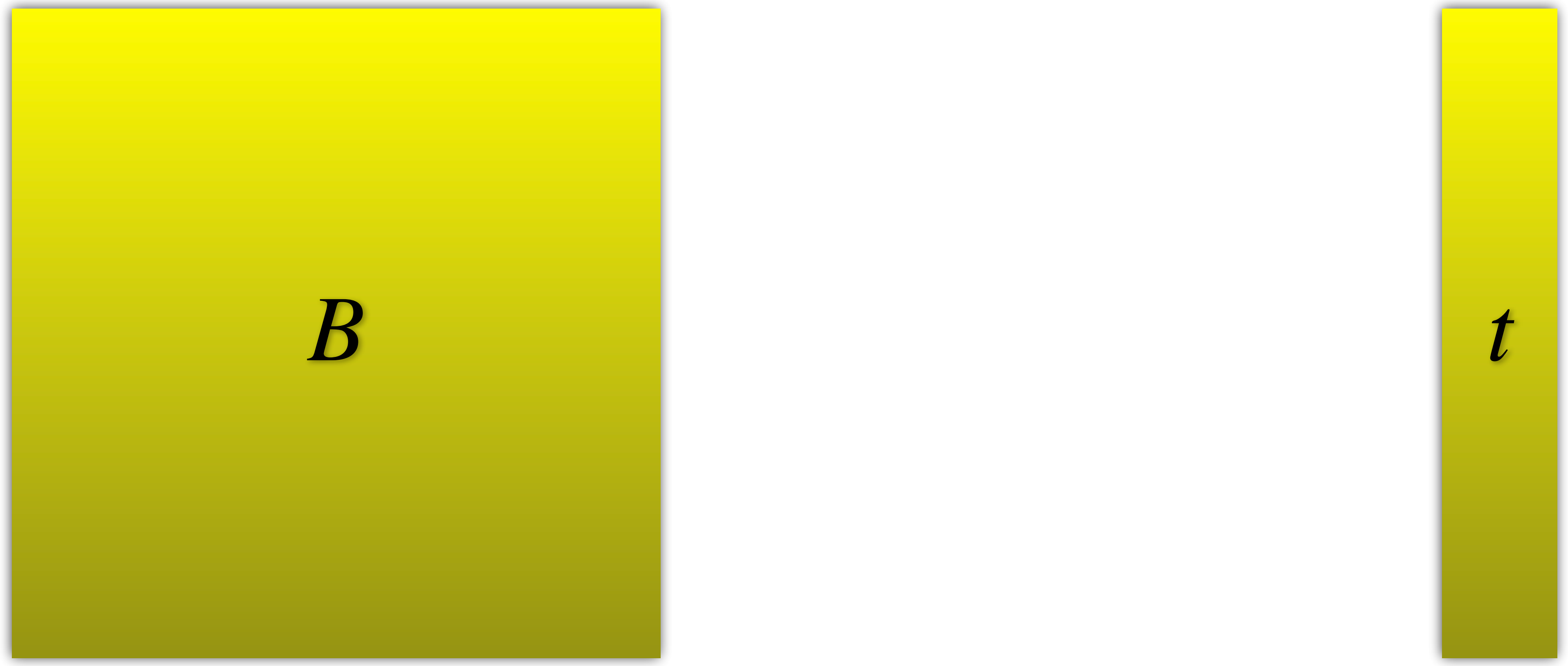

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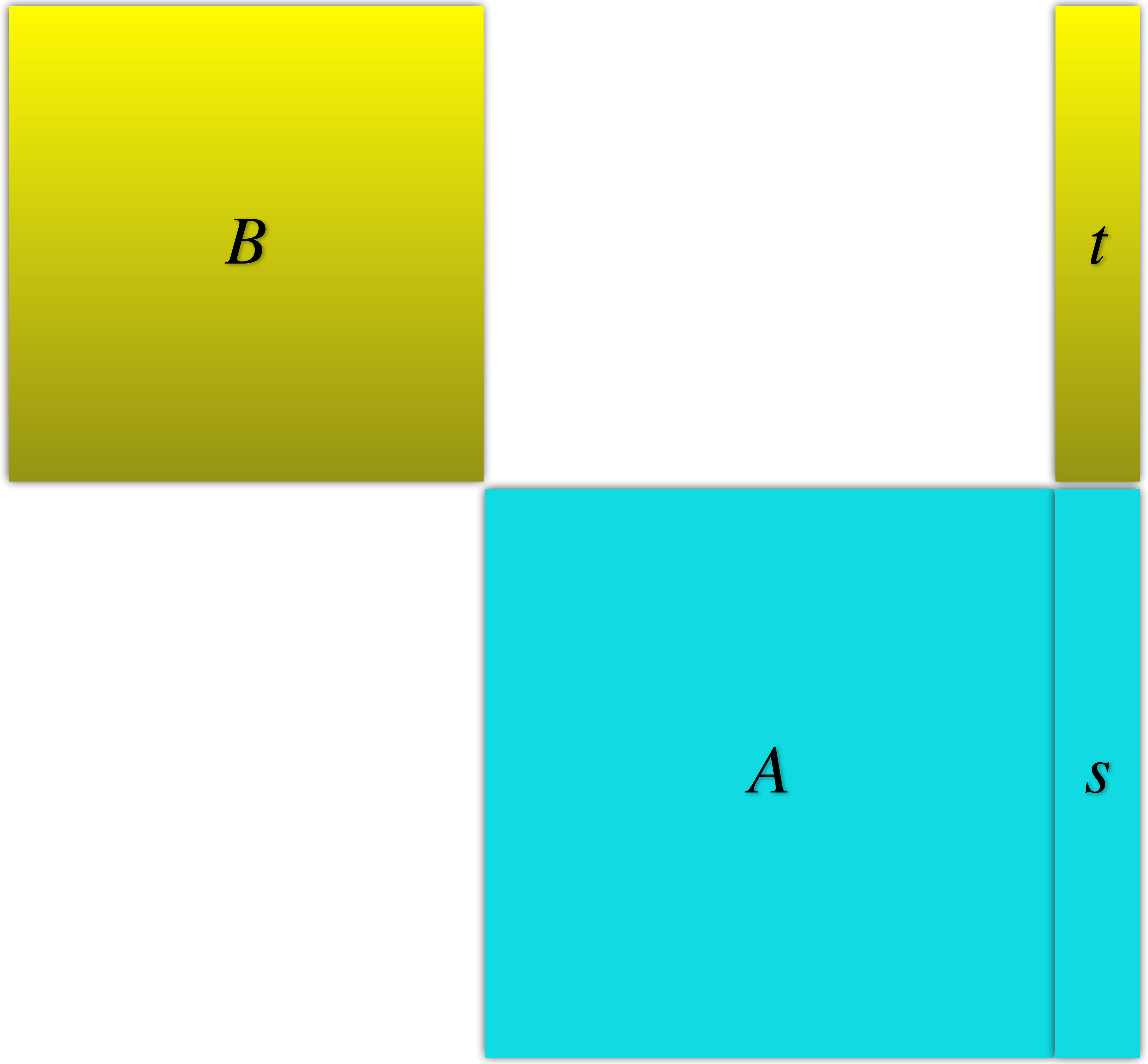

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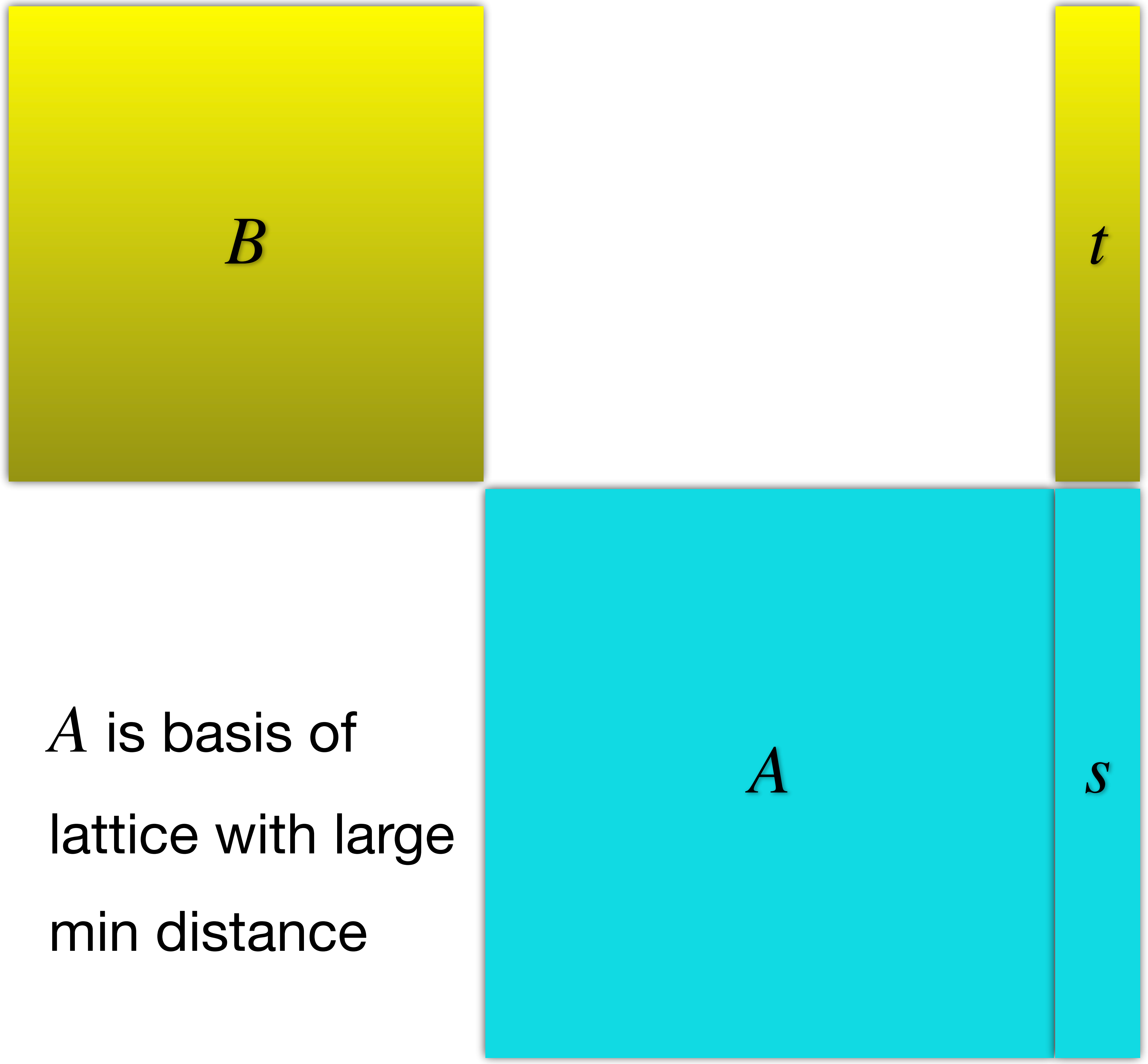

A is basis of lattice with large min distance

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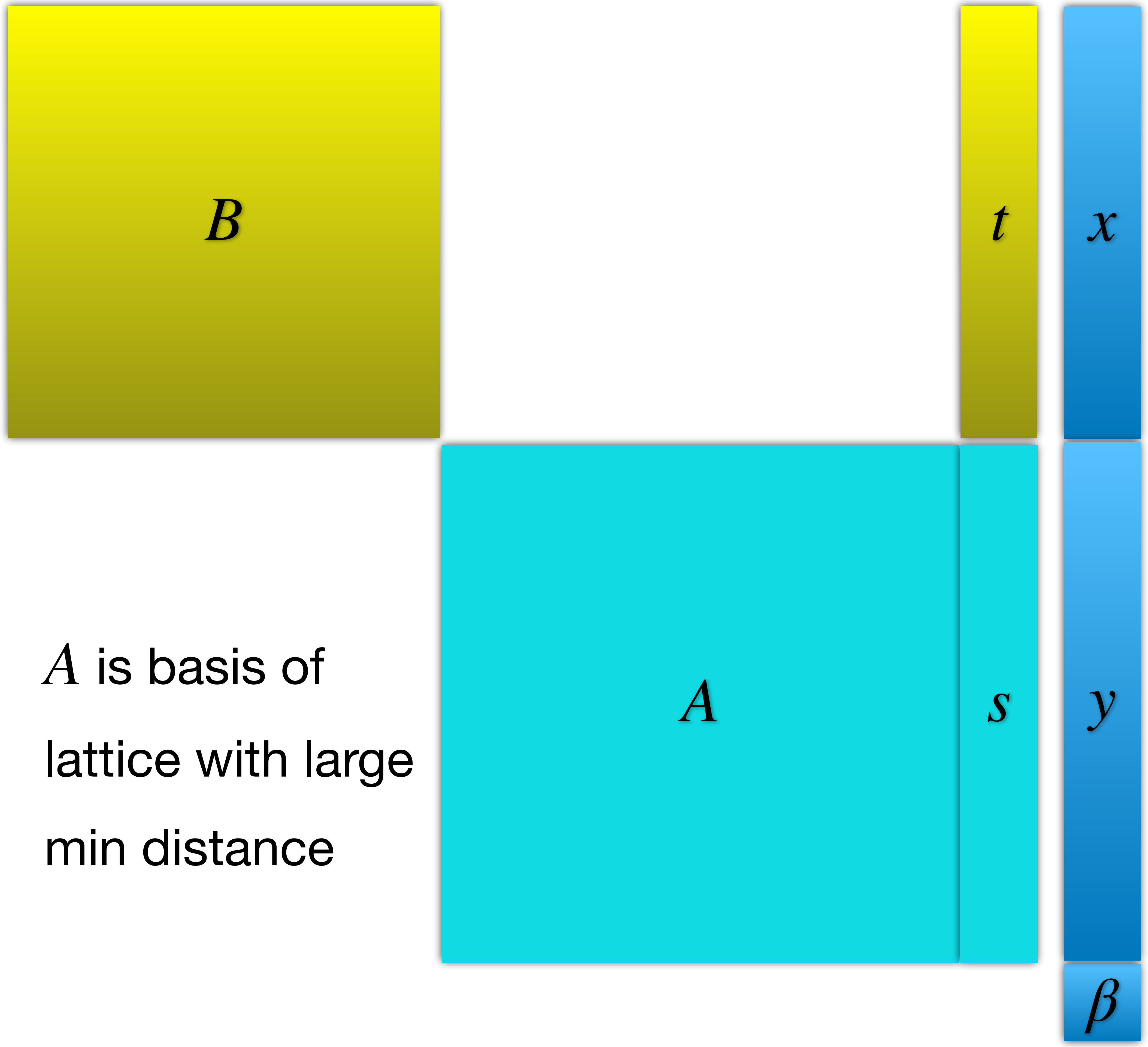

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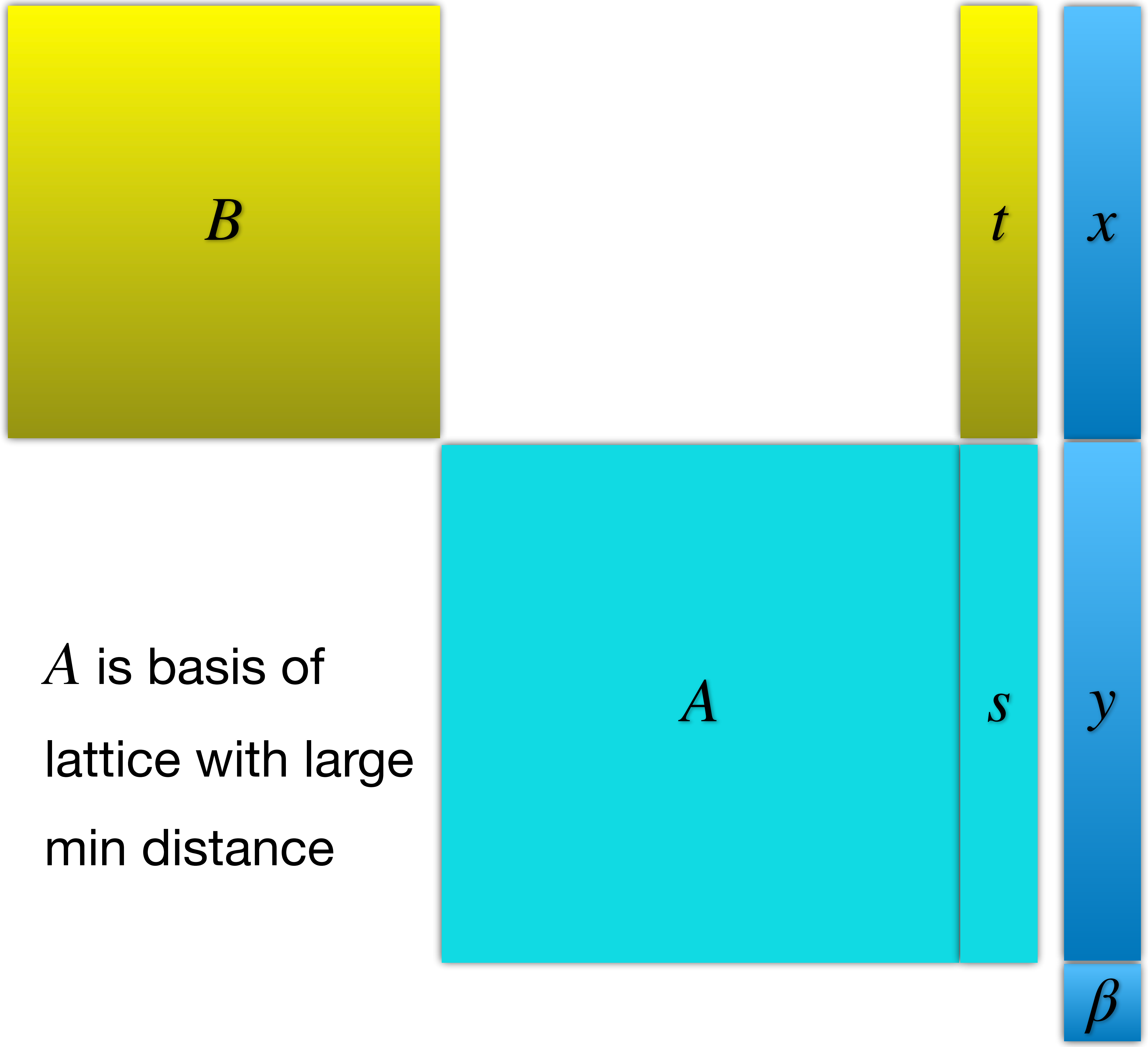

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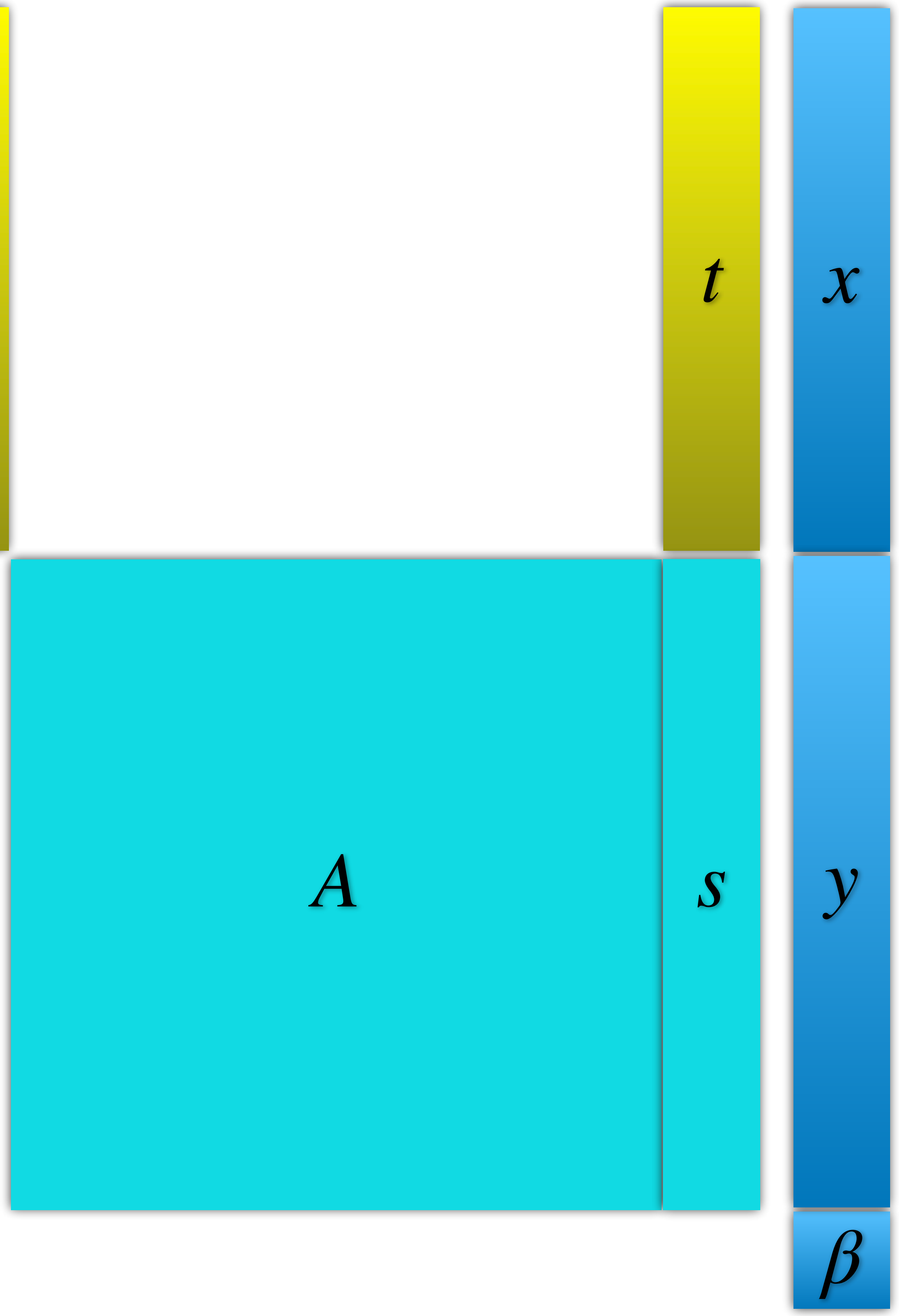

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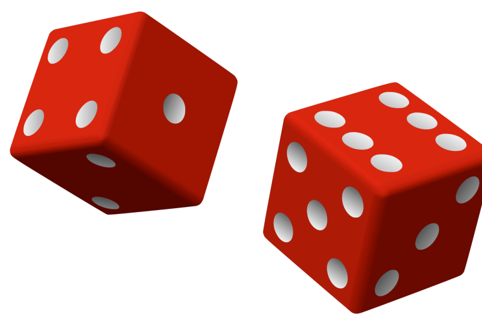
  $$N_{\text{good}} = |\{w \in L(A) : \|w - s\|_p \ll \gamma d\}|$$
- Provided that $N_{\text{good}} \gg N_{\text{bad}}$, obtain the desired SVP instance (B', d') by **randomly sparsifying** this intermediate lattice.



Locally dense lattices

Khot's reduction works if

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Locally dense lattice $L(A)$***:***

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A construction based on linear codes:

G generator matrix of binary BCH code with minimum distance $> \gamma d$:

$$L(A) = C(G) + 2\mathbb{Z}^m$$

with $m = \text{poly}(n)$.

Pros and cons of Khot's reduction

Pro: Khot's reduction from CVP to SVP is an FPT-reduction!

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Con2: Not easy to amplify approximation factor (we'll see more about this)

Insight 1: Khot for codes

Khot's approach also reduces γ-NCP$_q$ to γ'-MDP$_q$ for some $\gamma(q), \gamma'(q) > 1$!

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How can we get $W[1]$-hardness for **all** $\gamma' > 1$?

Amplifying the approximation factor in coding problems

γ-MDP$_q$ is $W[1]$-hard for **some** approximation factor $\gamma' > 1$ $\implies$ γ-MDP$_q$ also $W[1]$-hard for $\gamma' > \gamma$

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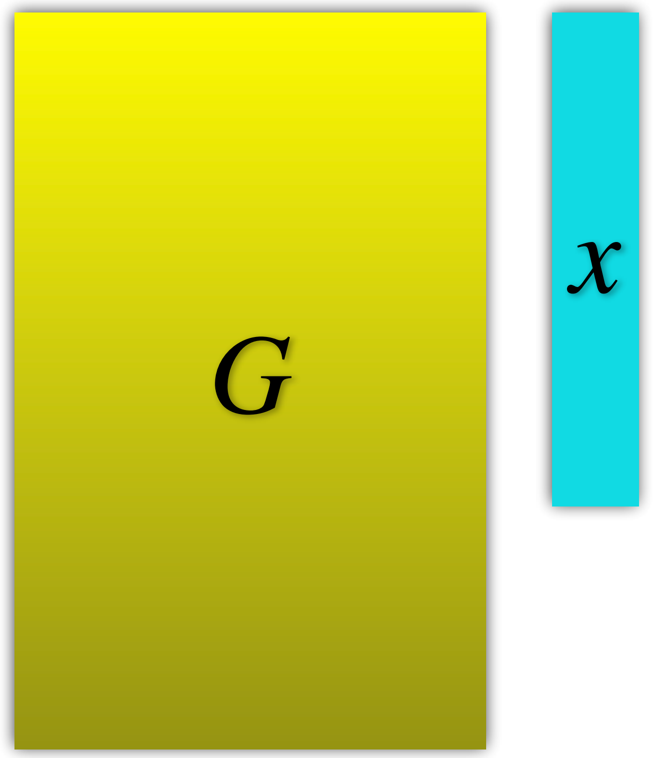

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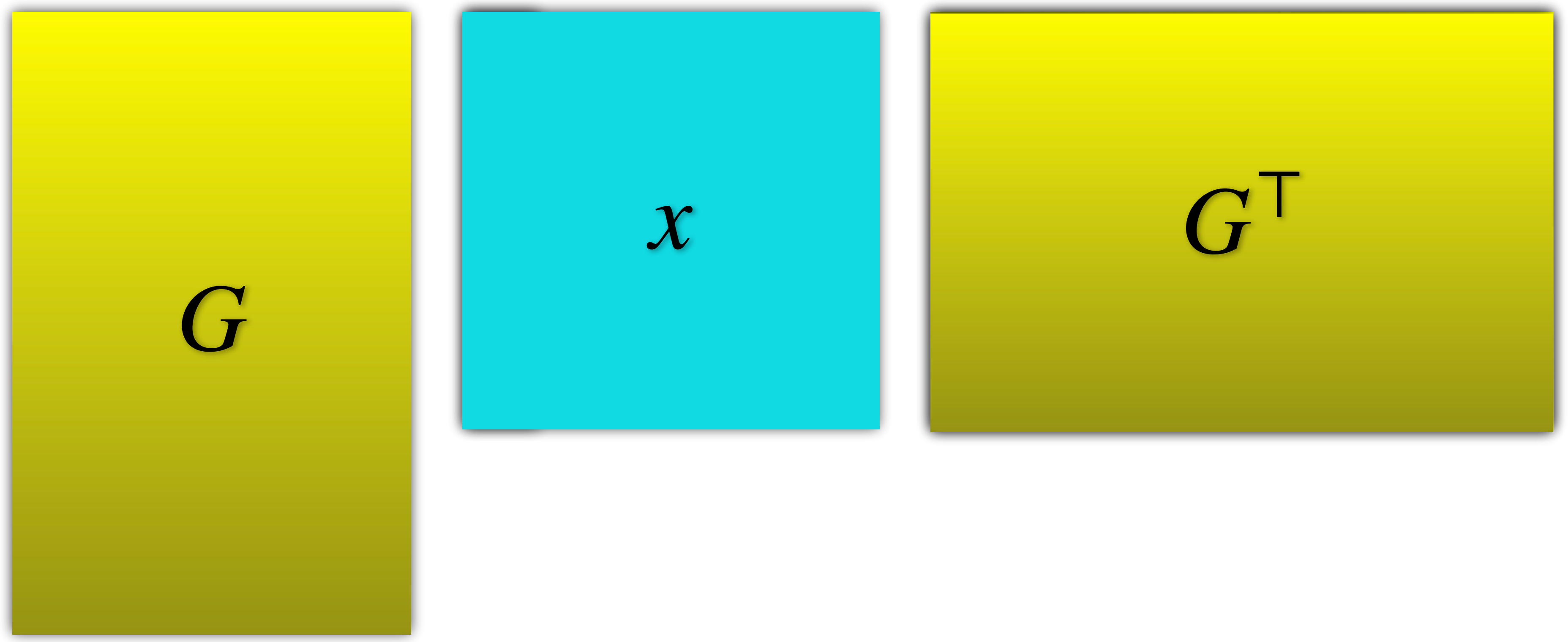

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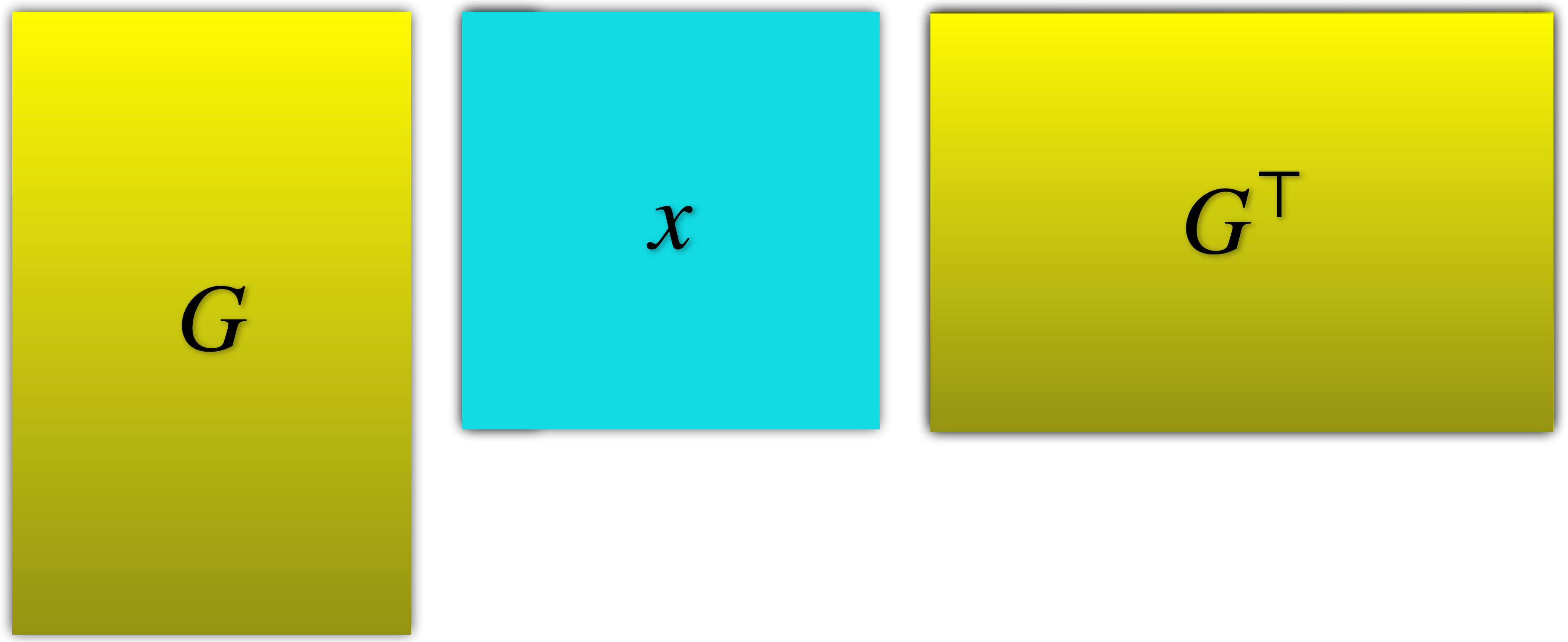

Use the fact that $d(C \otimes C) = d(C)^2$:

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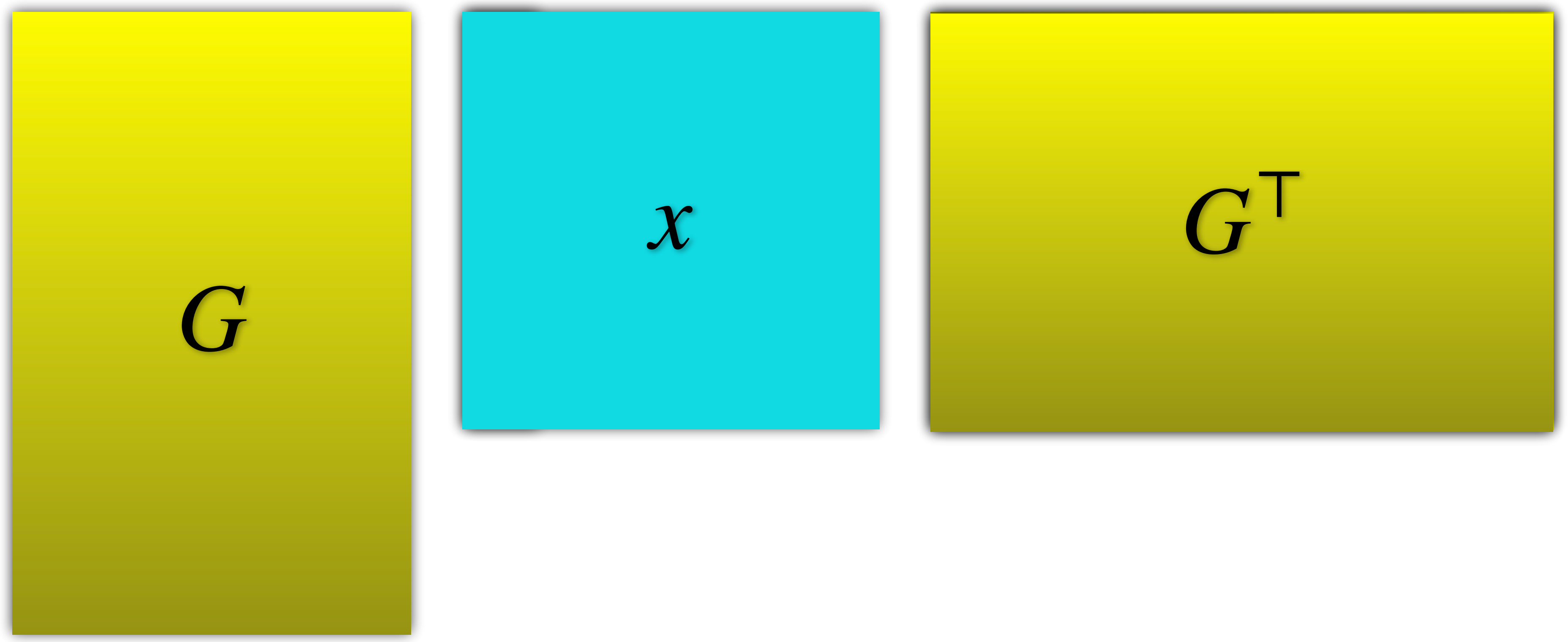

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Conclusion: γ-SVP$_1$ is $W[1]$**-hard** for any approximation factor $\gamma < 2$.

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"Hmm… Cool problem!"

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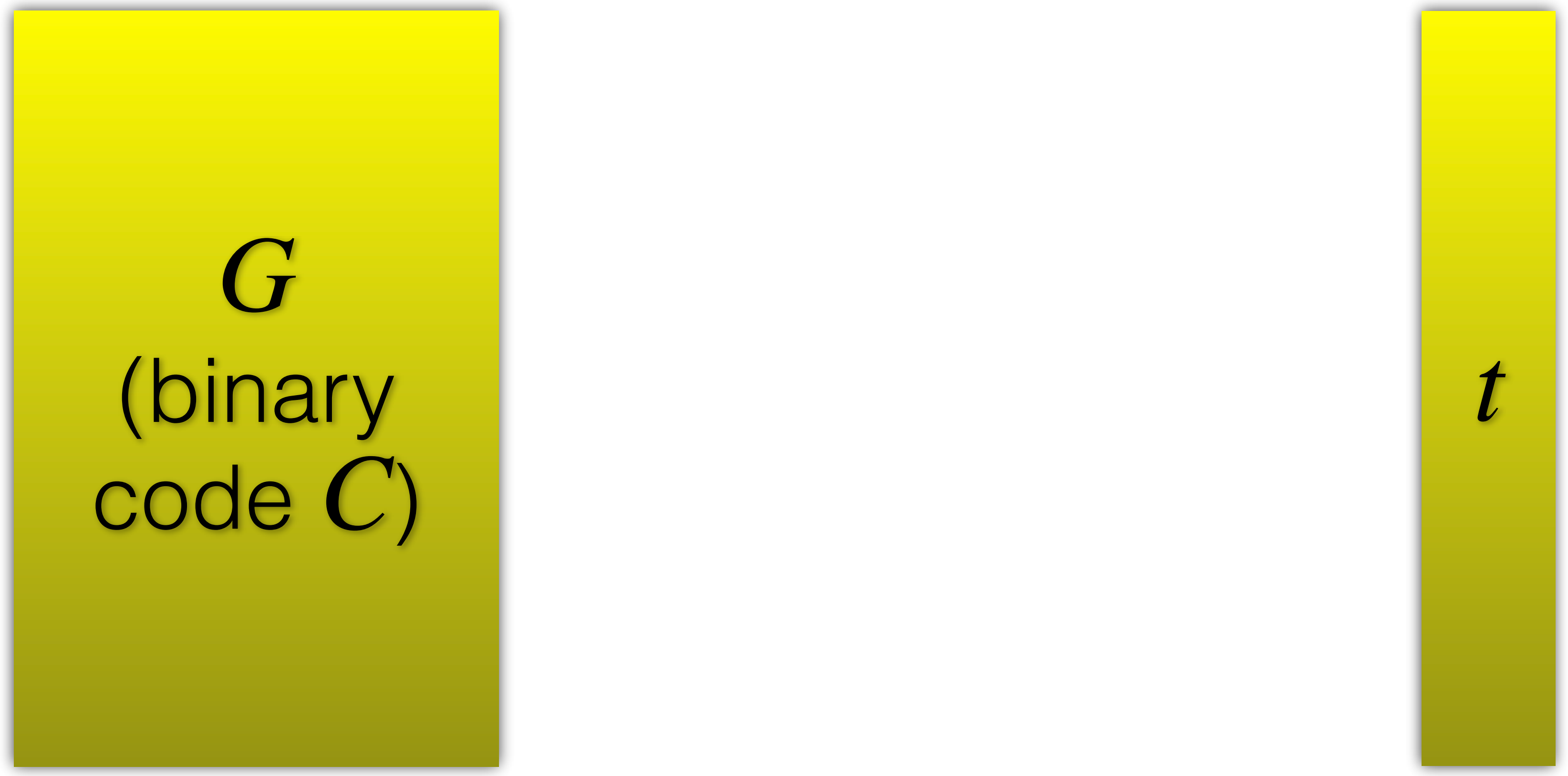

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(construction A)

$2t$

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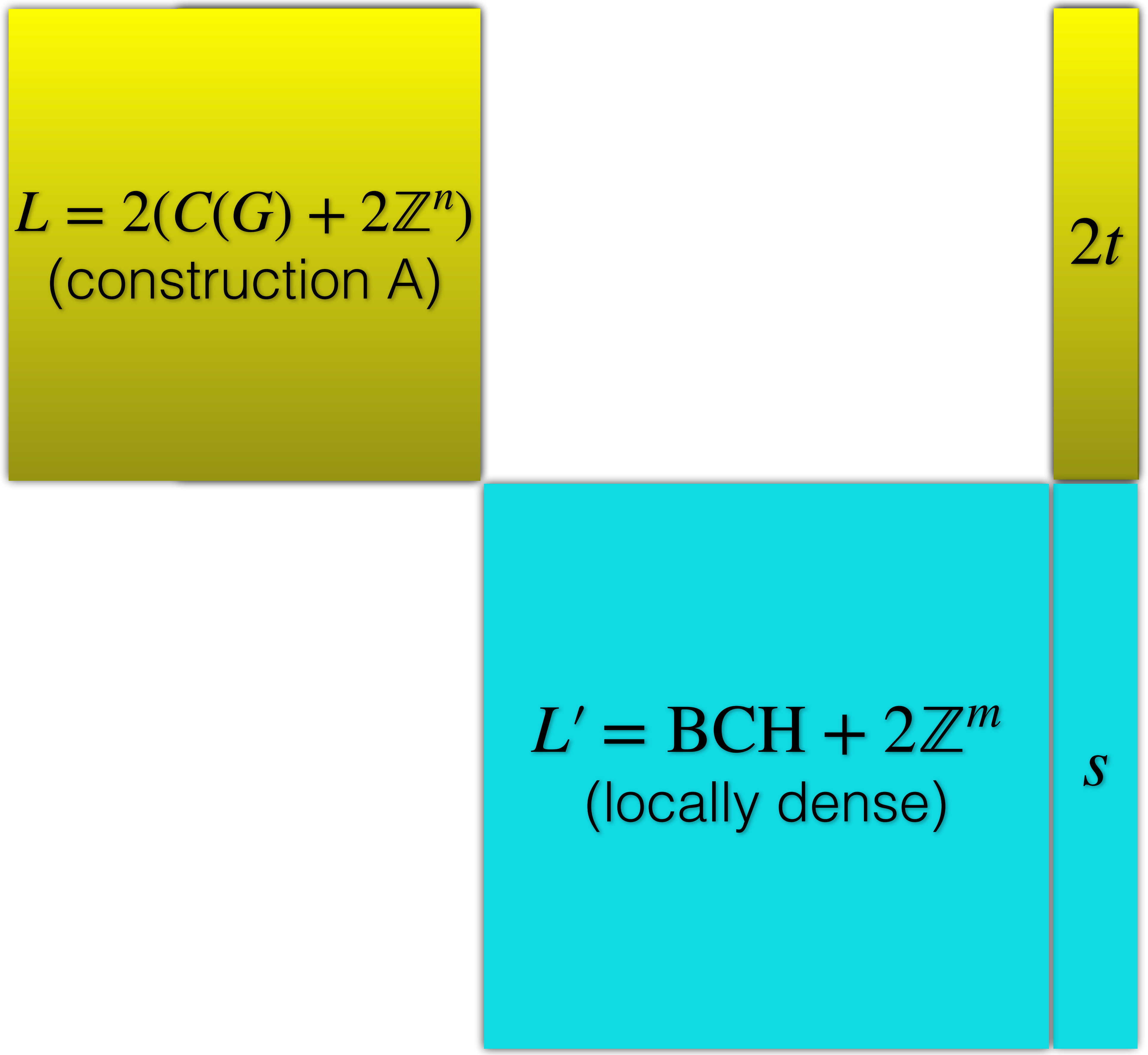


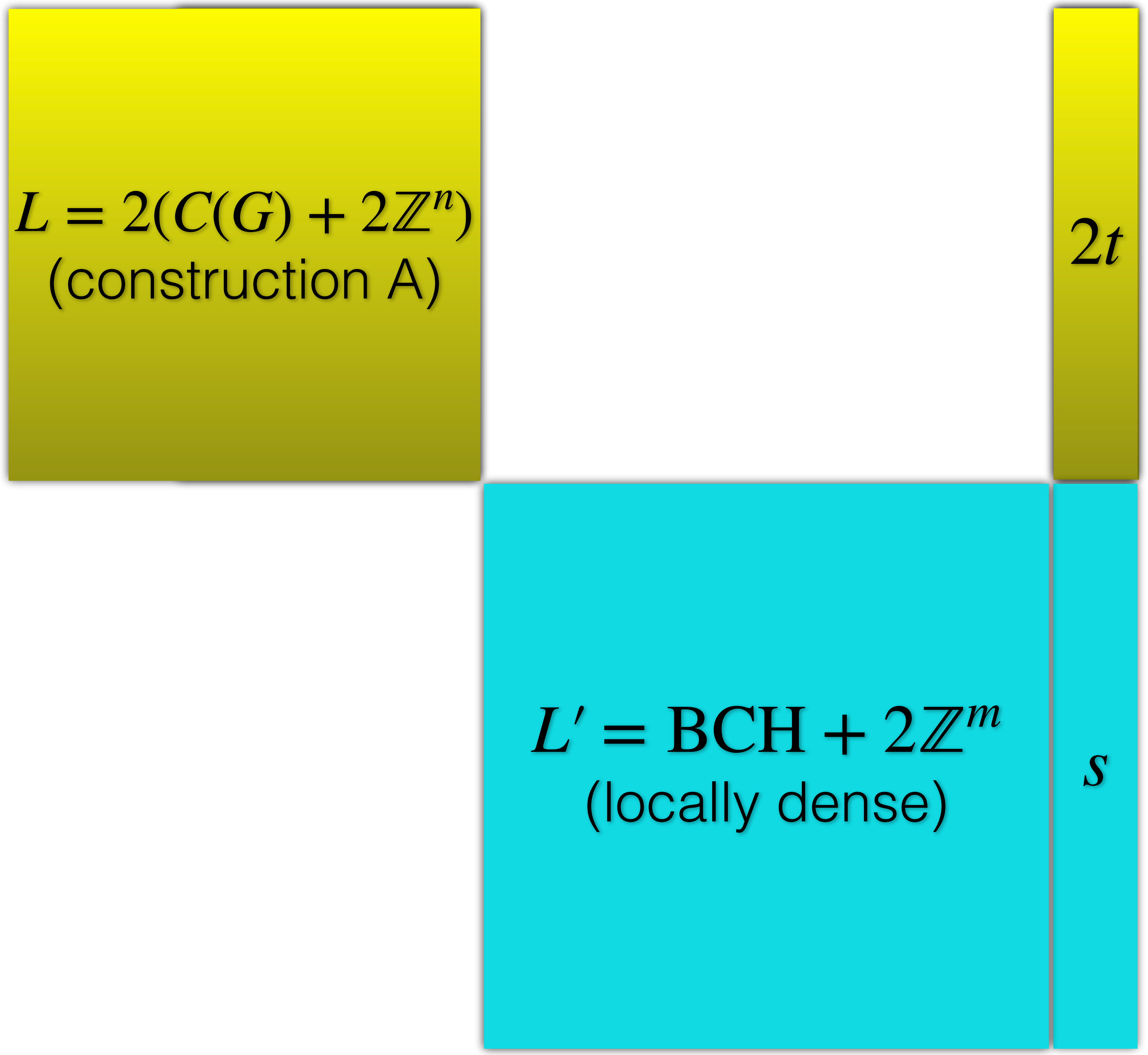
Insight 3: Turning codes into lattices

Need a different approach to amplify the approximation factor of SVP!

Idea: Apply Khot's approach to reduce from NCP_2 to SVP_p !

Can ensure Haviv-Regev tensoring conditions are satisfied in this case.

Conclusion: γ-SVP_p is $W[1]$**-hard** for all $p > 1$ and **all approx factor** $\gamma > 1$.



Wrapping up

- Deterministic FPT reductions?
  Unknown for SVP even in the unparameterized setting!

- $W[1]$-hardness of γ-SVP$_1$ for all approximation factors $\gamma > 1$?
  The last missing puzzle piece!

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Thanks!