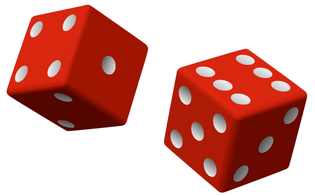
Sampling codes with memory and the insertion list-decoding capacity



João Ribeiro

Inst. Telecomunicações + Técnico-ULisboa

Based on joint work with

Roni Con

Technion → Tel Aviv U

Dean Doron

Ben-Gurion U

Funded by the European Union

European Research Council

Established by the European Commission

Insertions and deletions

Simple models of errors that cause loss of synchronization.

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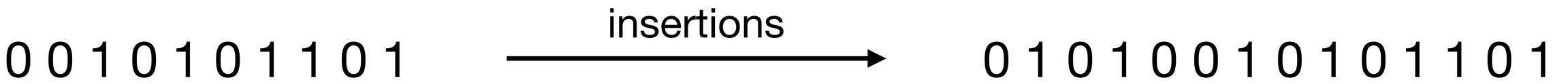

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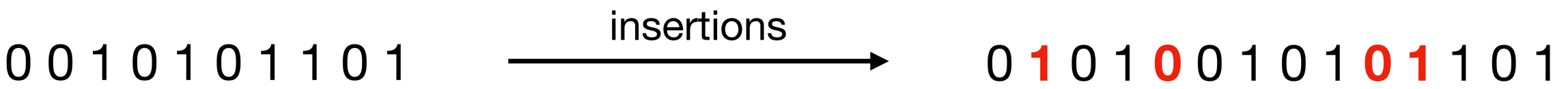

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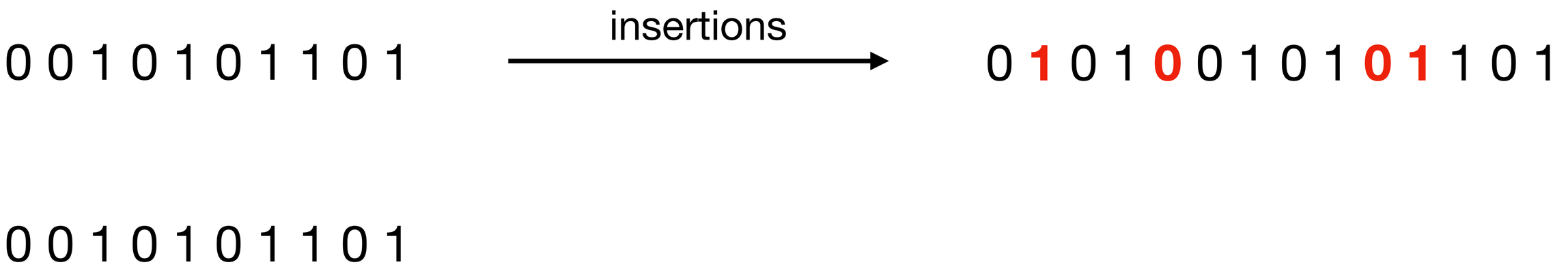

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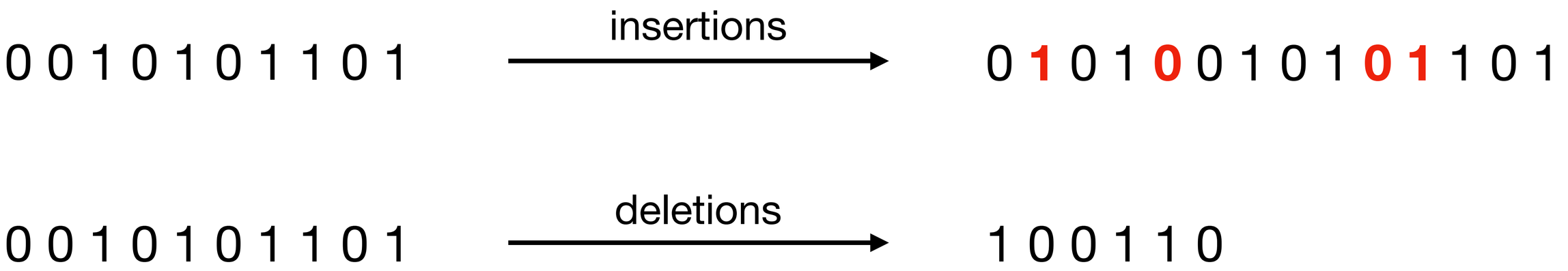

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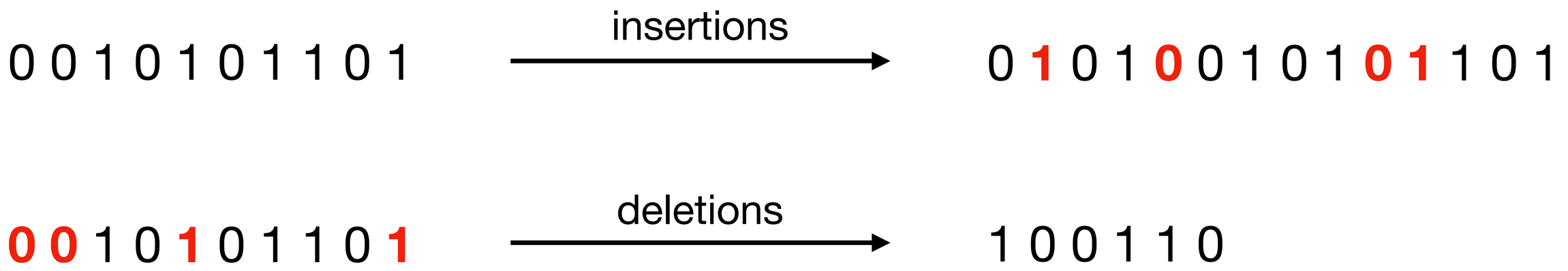

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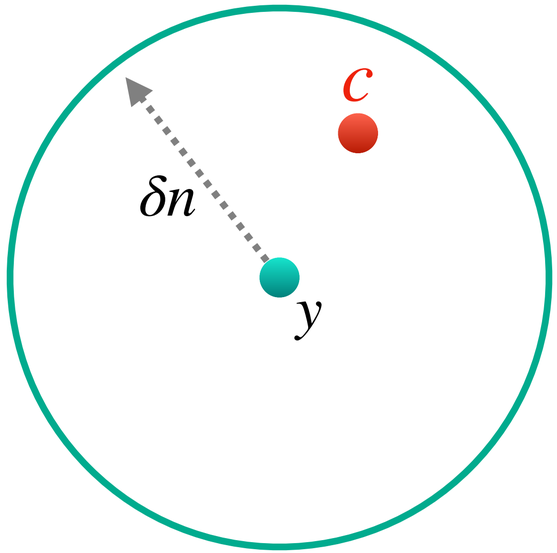

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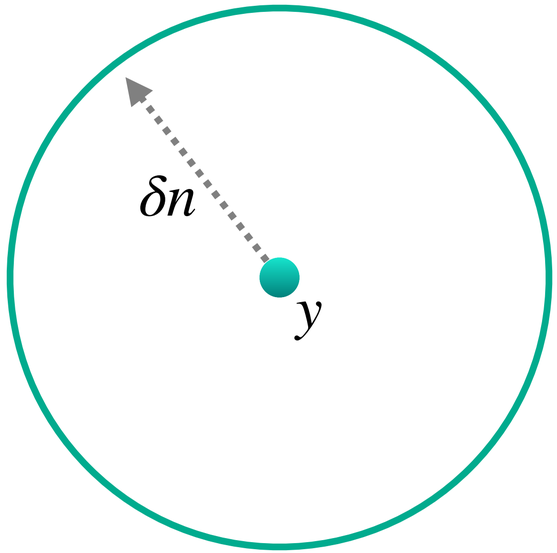

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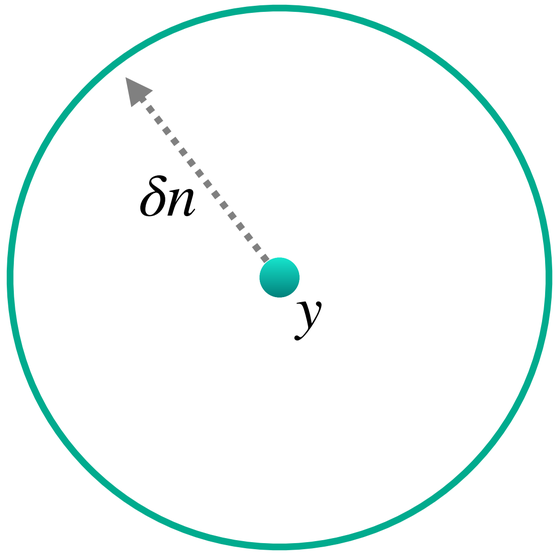

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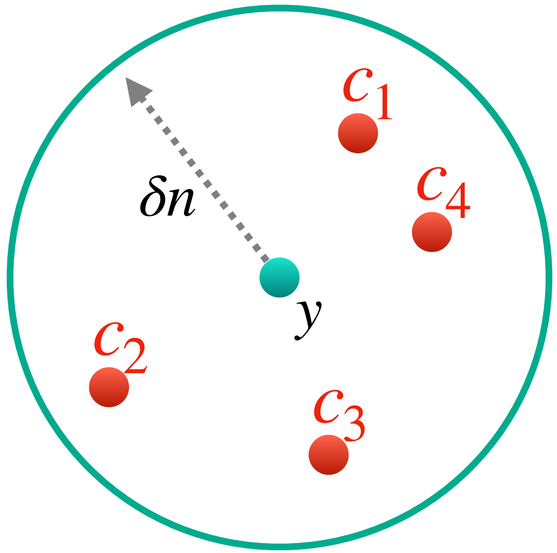

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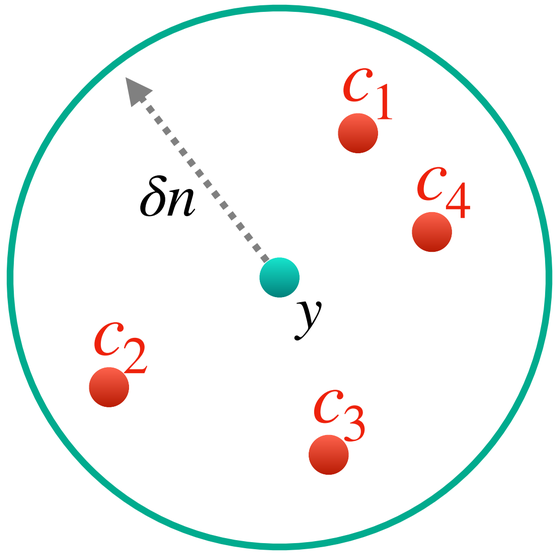

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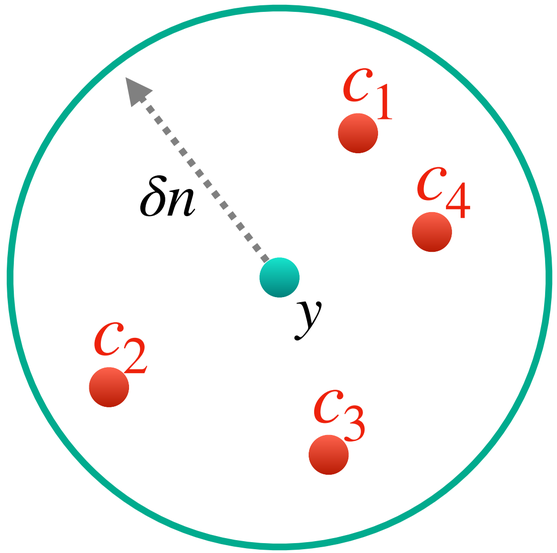

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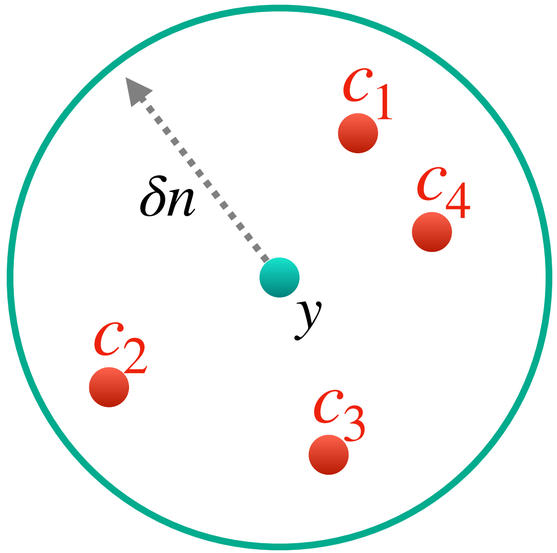

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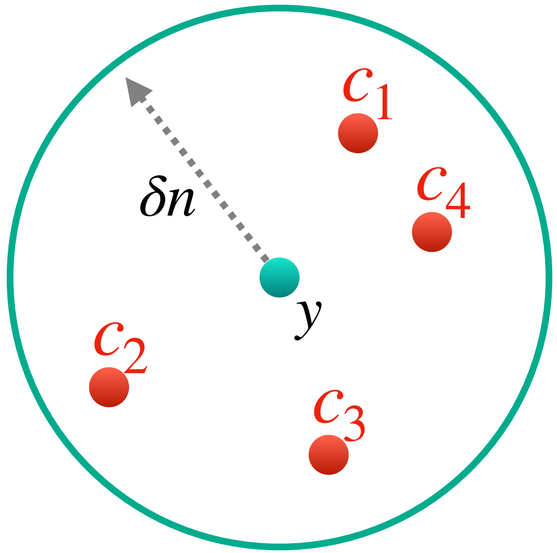

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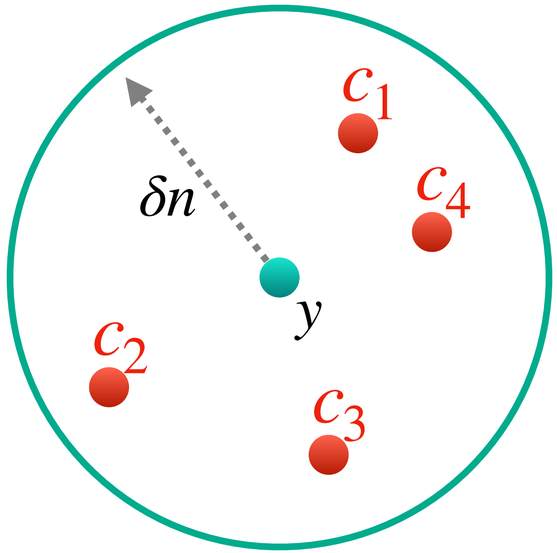

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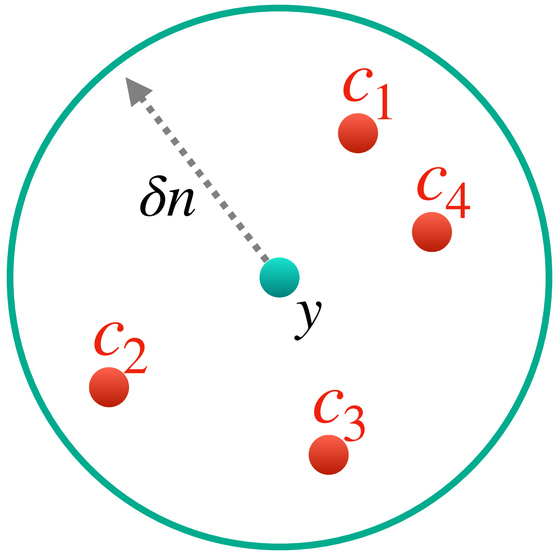

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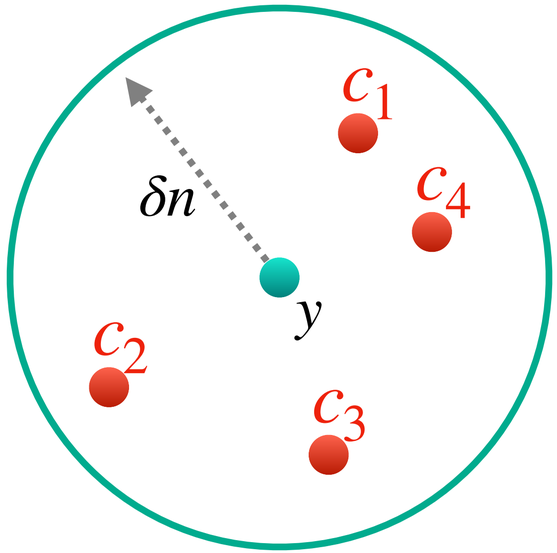

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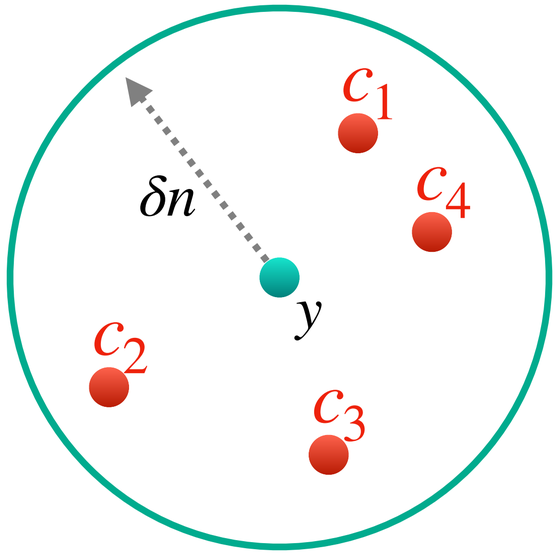

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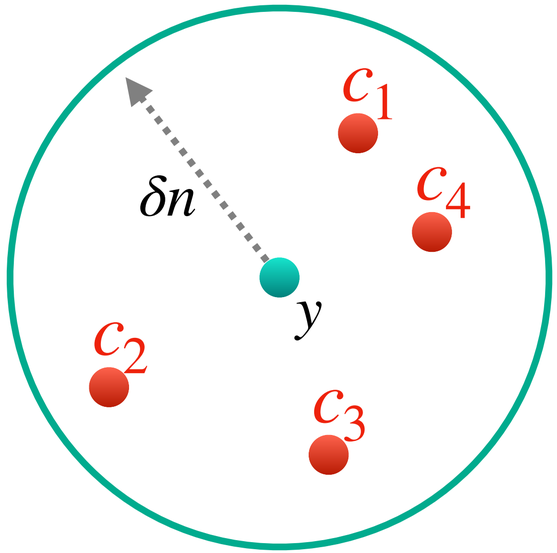

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Analogous question for unique decoding ($L = 1$) is a major open problem in coding theory, even for bit-flips!

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This matches the capacity of the binary symmetric channel. (Not a coincidence!)

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This is < 1 when $L \geq 1/\varepsilon$.

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- At rate $R = 1 - h(\delta) - \varepsilon$ we can achieve list size $1/\varepsilon$ with a uniformly random code;
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Much less is known!

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with constant list size [Haeupler-Shahrasbi-Sudan '18]*

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intricate explicit construction
[Guruswami-Haeupler-Shahrasbi '20]

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+ Bukh-Guruswami-Håstad
[Wachter-Zeh '17, Hayashi-Yasunaga '18, Liu-Tjuawinata-Xing '23, Liang '26]

$\Longrightarrow$ $C_{\text{ins}}(\delta) > 0$ for all $\delta < 0.707$.

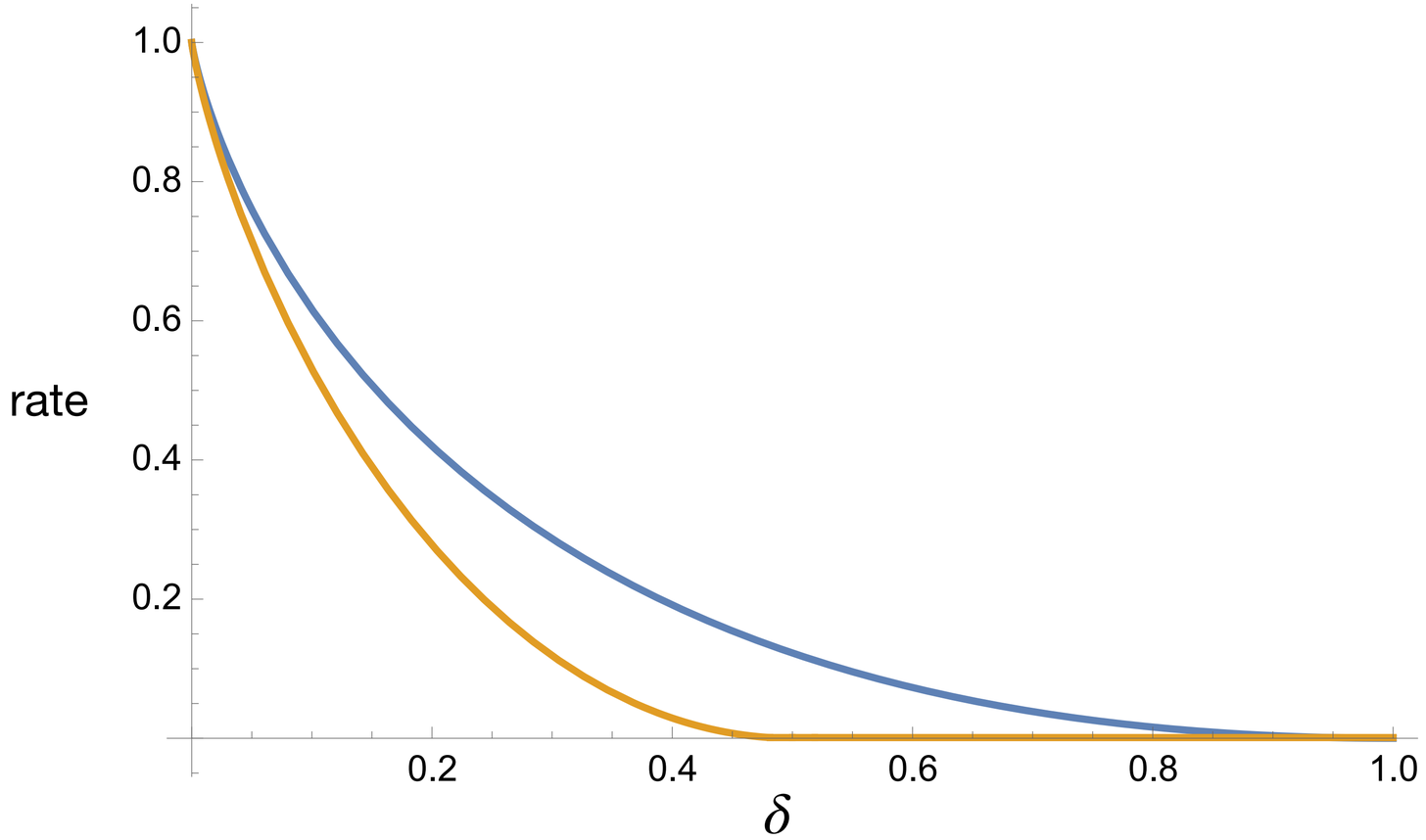
intricate explicit construction
[Guruswami-Haeupler-Shahrasbi '20]

$\Longrightarrow$ $C_{\text{ins}}(\delta) > 0$ for all $\delta < 1$.

In both cases, the lower bound is tiny!

Capacity of list-decoding from insertions — the story so far

Summarizing:



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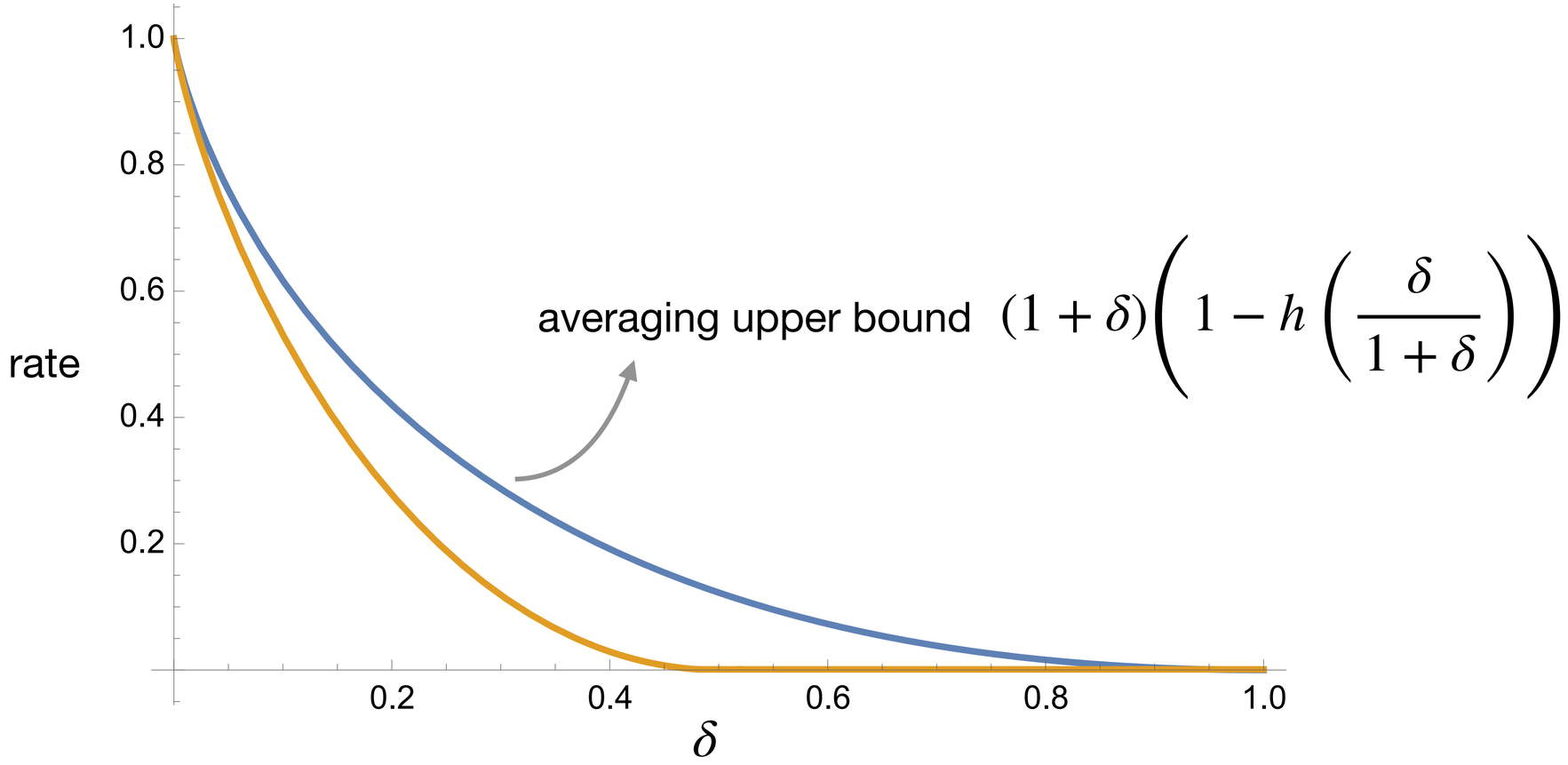

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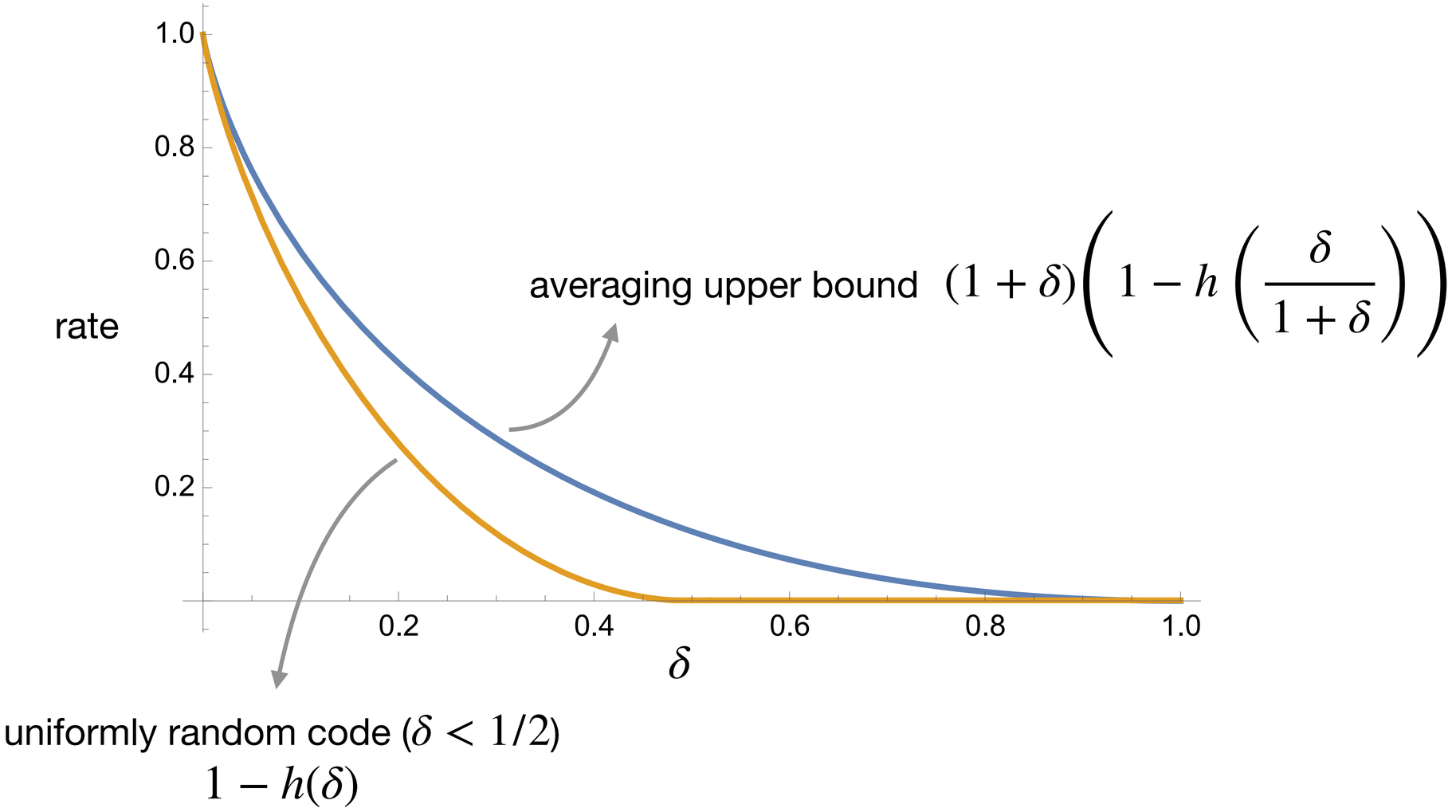

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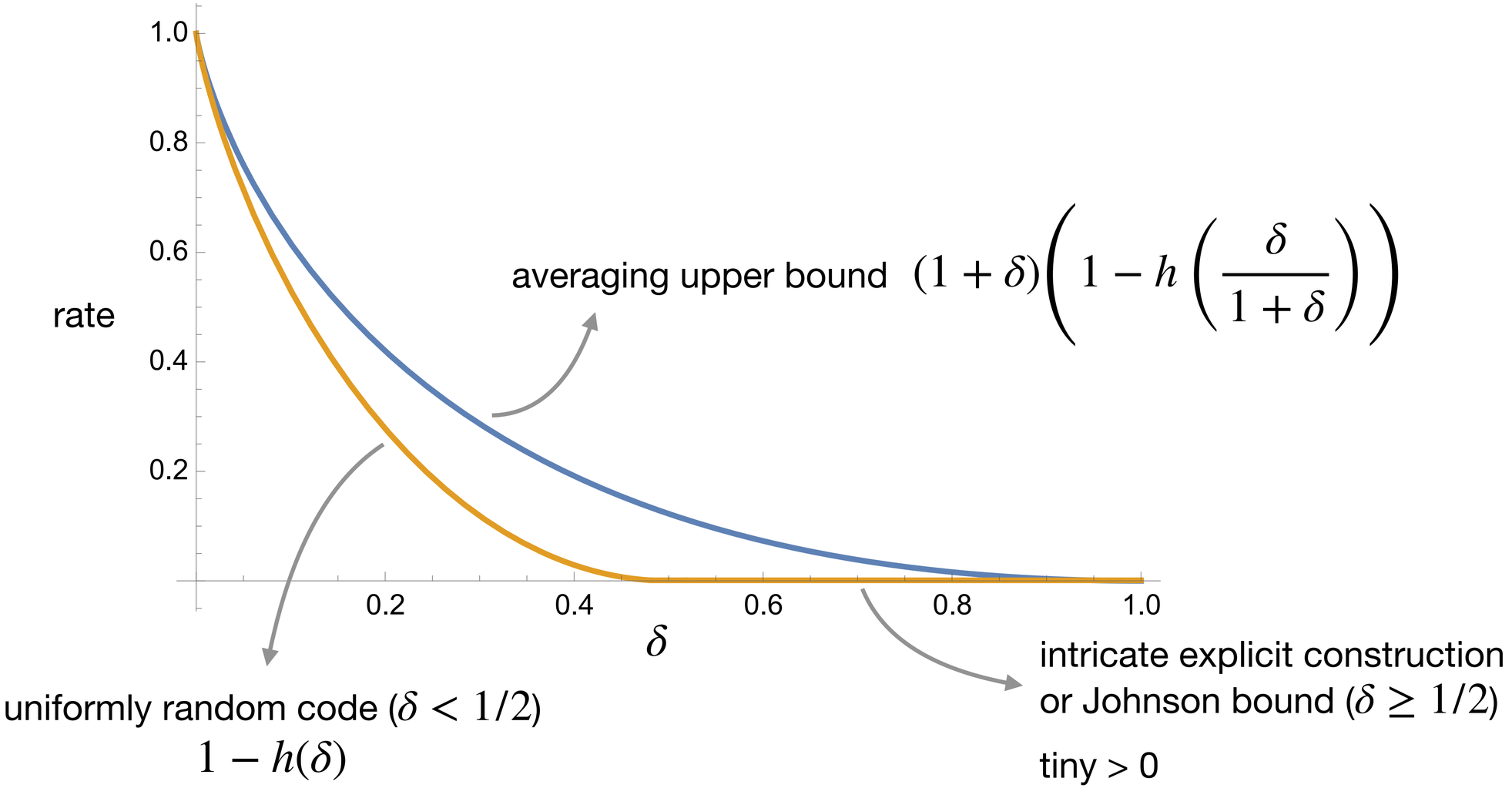

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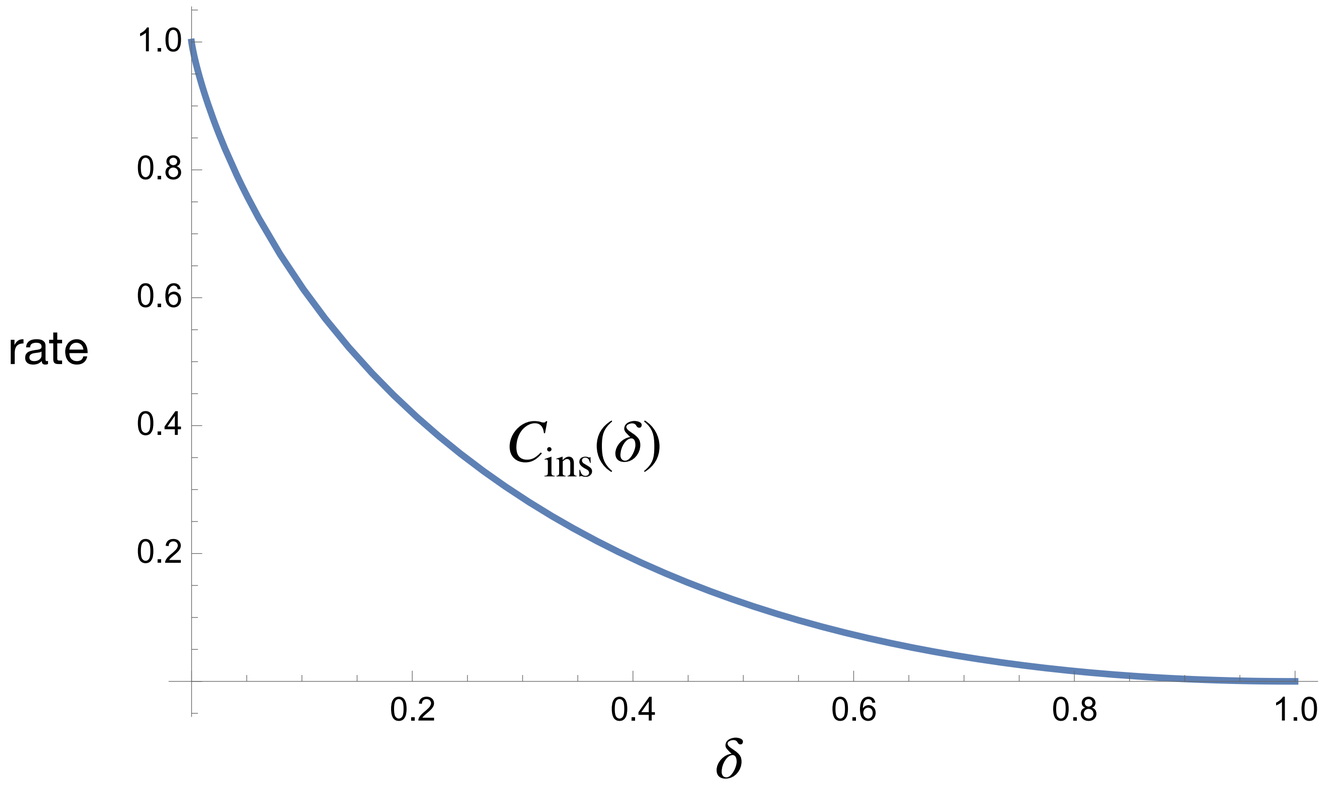

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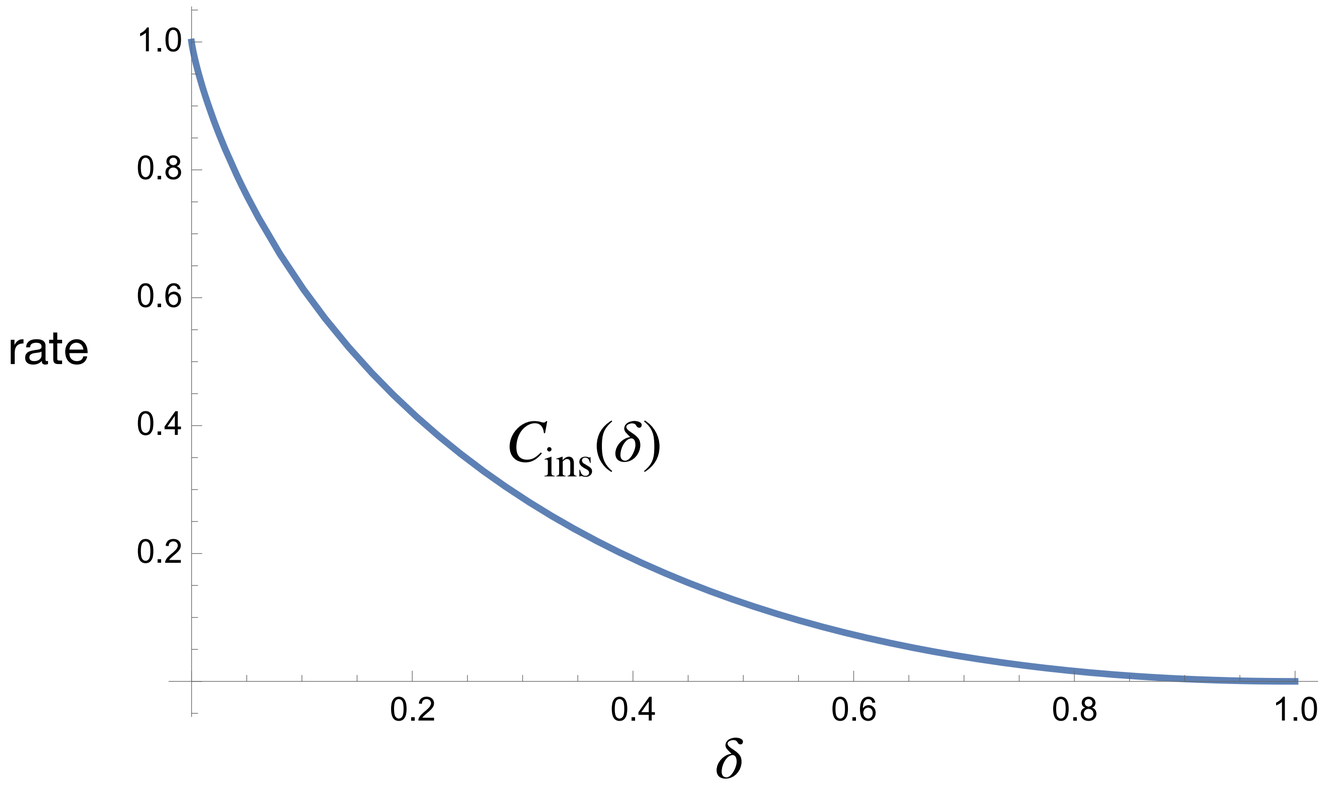

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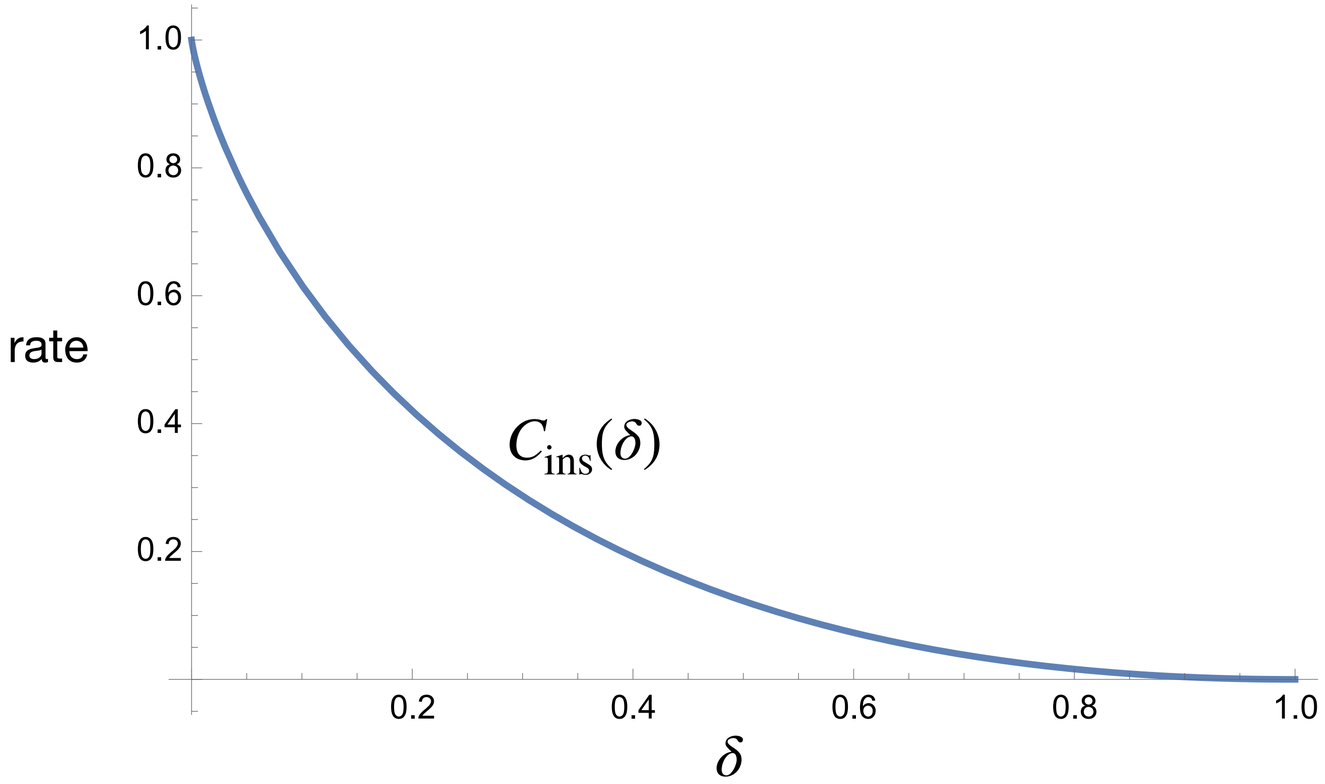

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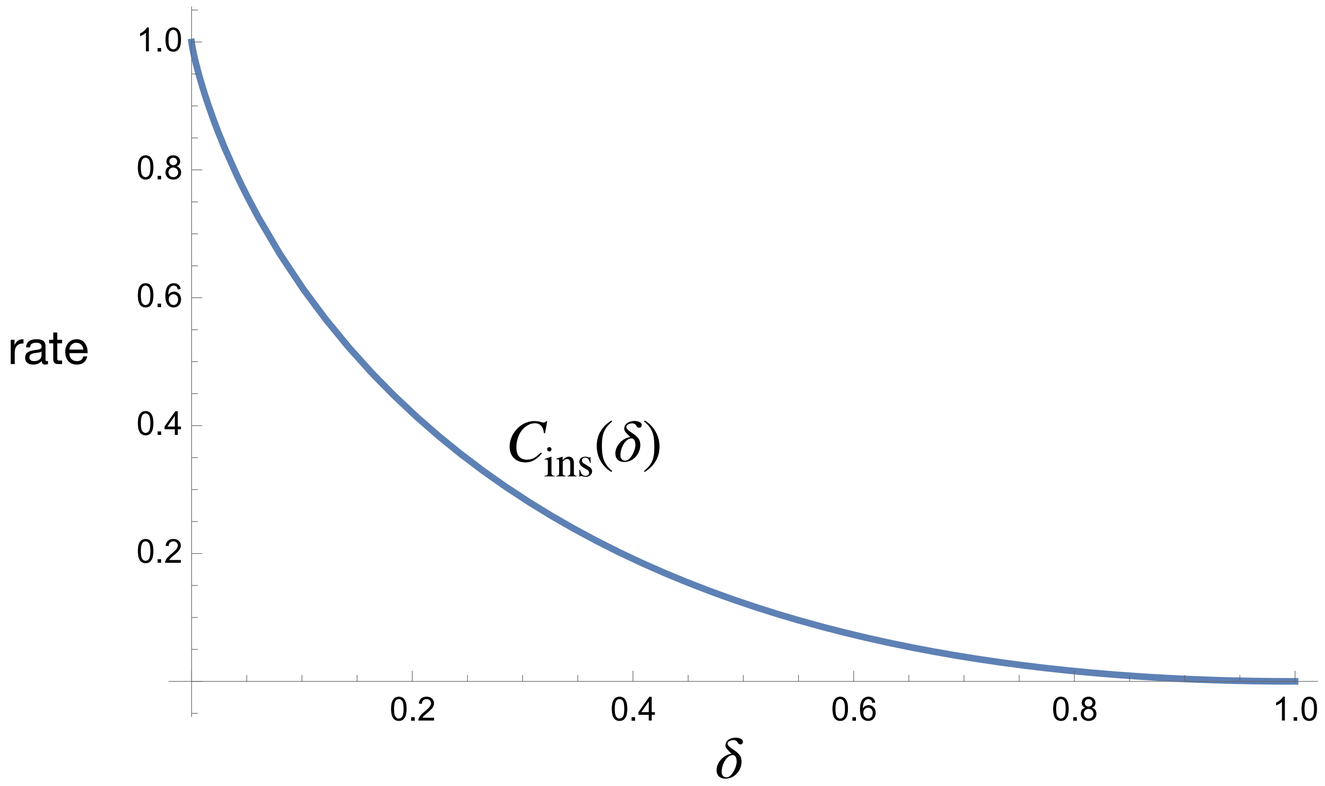

We actually prove something stronger:

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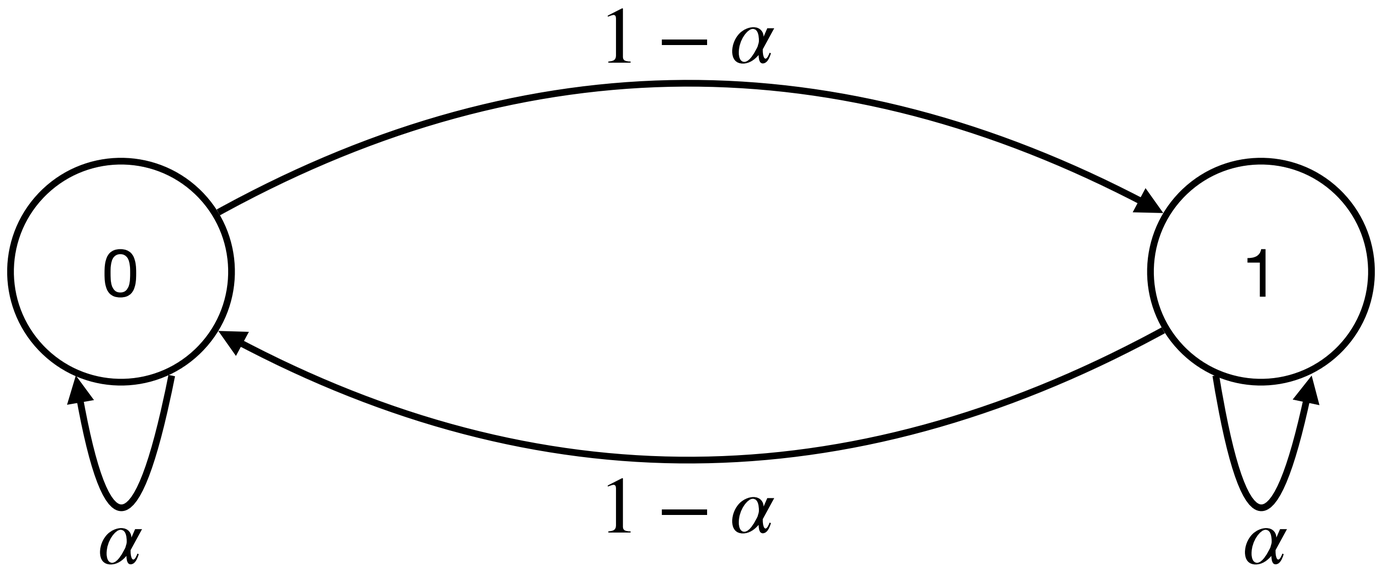

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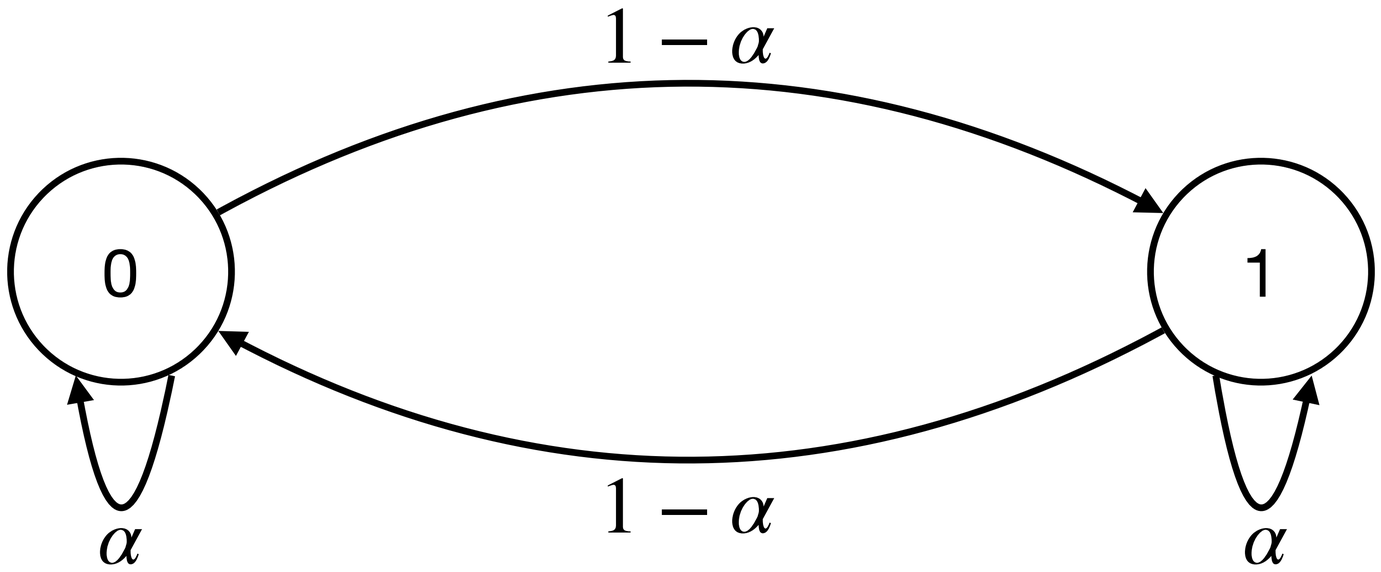

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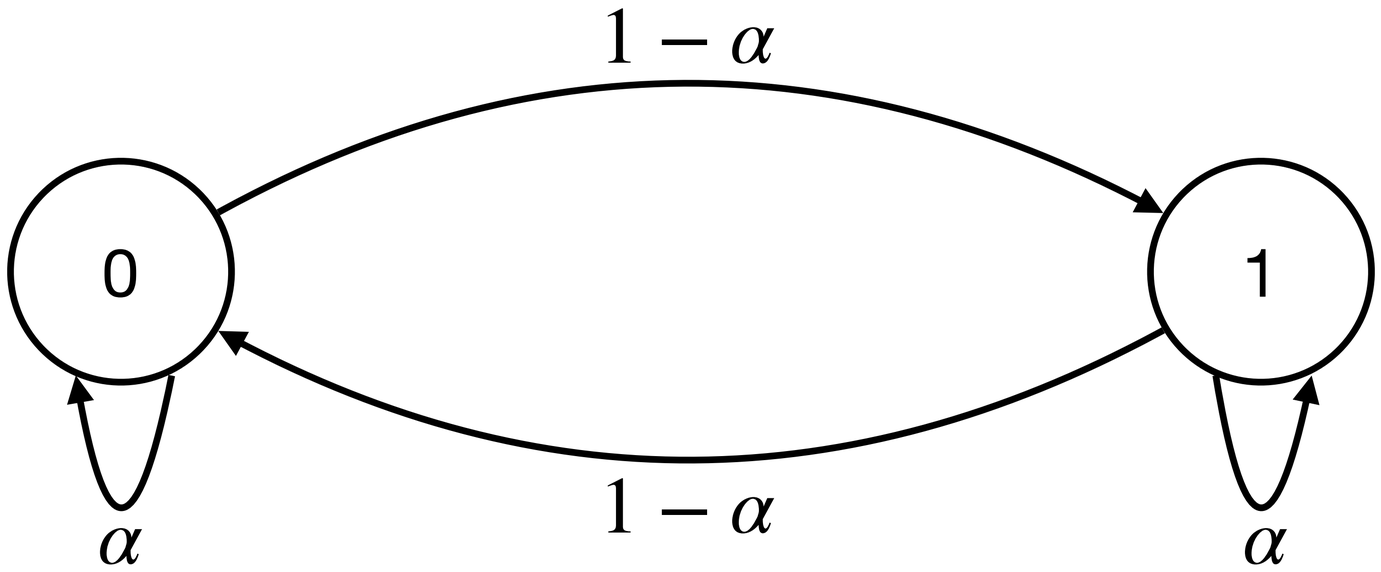

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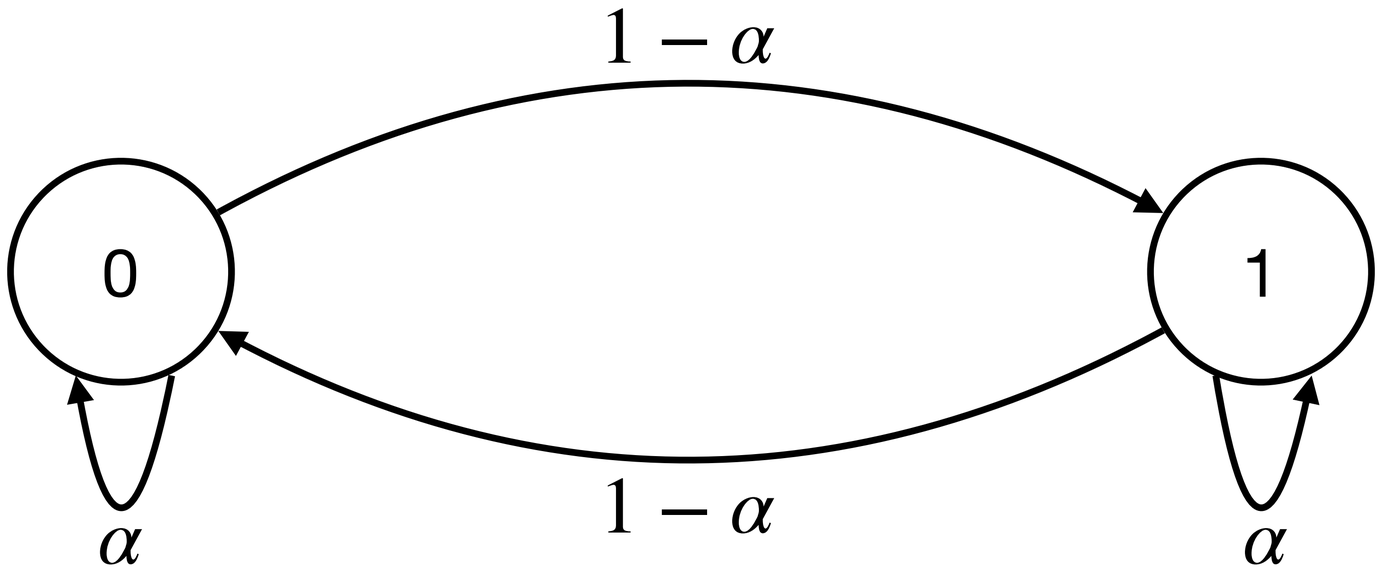

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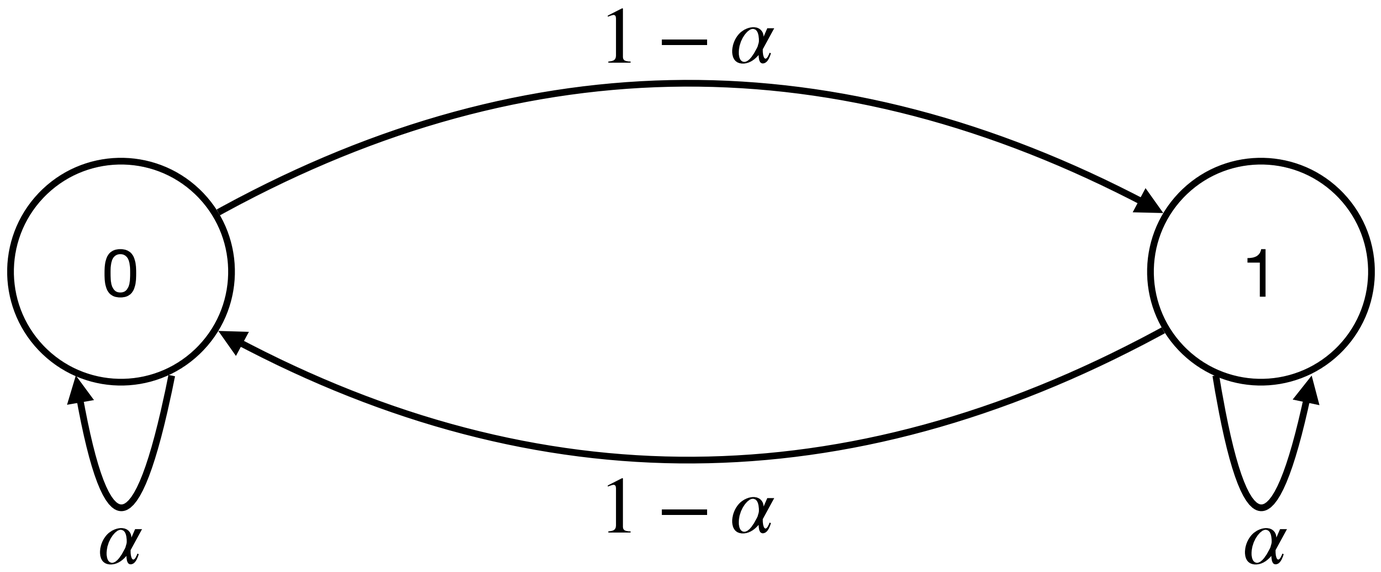
Choosing $\alpha = 1/2$ recovers uniformly random coding.



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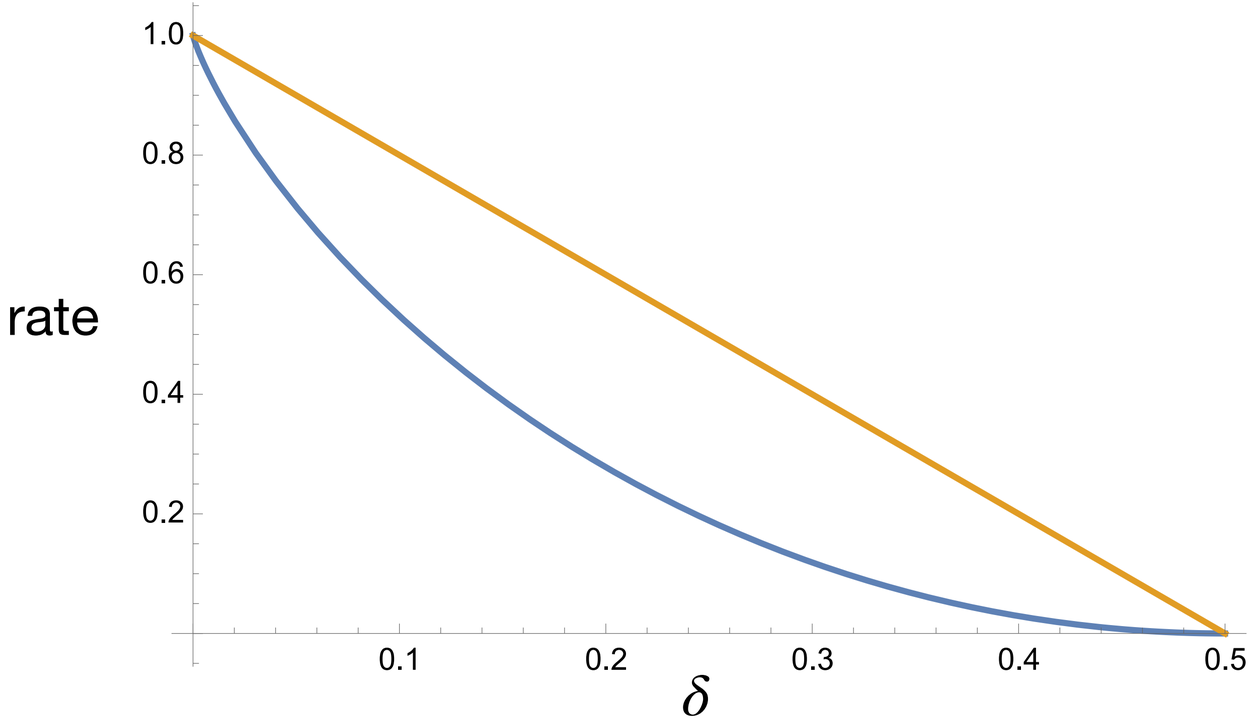

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$\Longrightarrow$ **Markov codes never do better than random!**

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Perhaps you have good ideas. :)

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What’s going on??

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Sharp upper bound for small deletion rate δ:

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- Uniformly random coding is essentially optimal for small δ;
- Matches asymptotic behavior of deletion channel capacity!

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Idea: If C has high rate, then most codewords of C must have many runs. And strings with many runs have many subsequences. So the list size must be large.

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take $\gamma = \frac{1}{2} - \sqrt{\frac{h(\delta)\ln 2}{2}}$

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