Discrepancy Problems for Linear Codes

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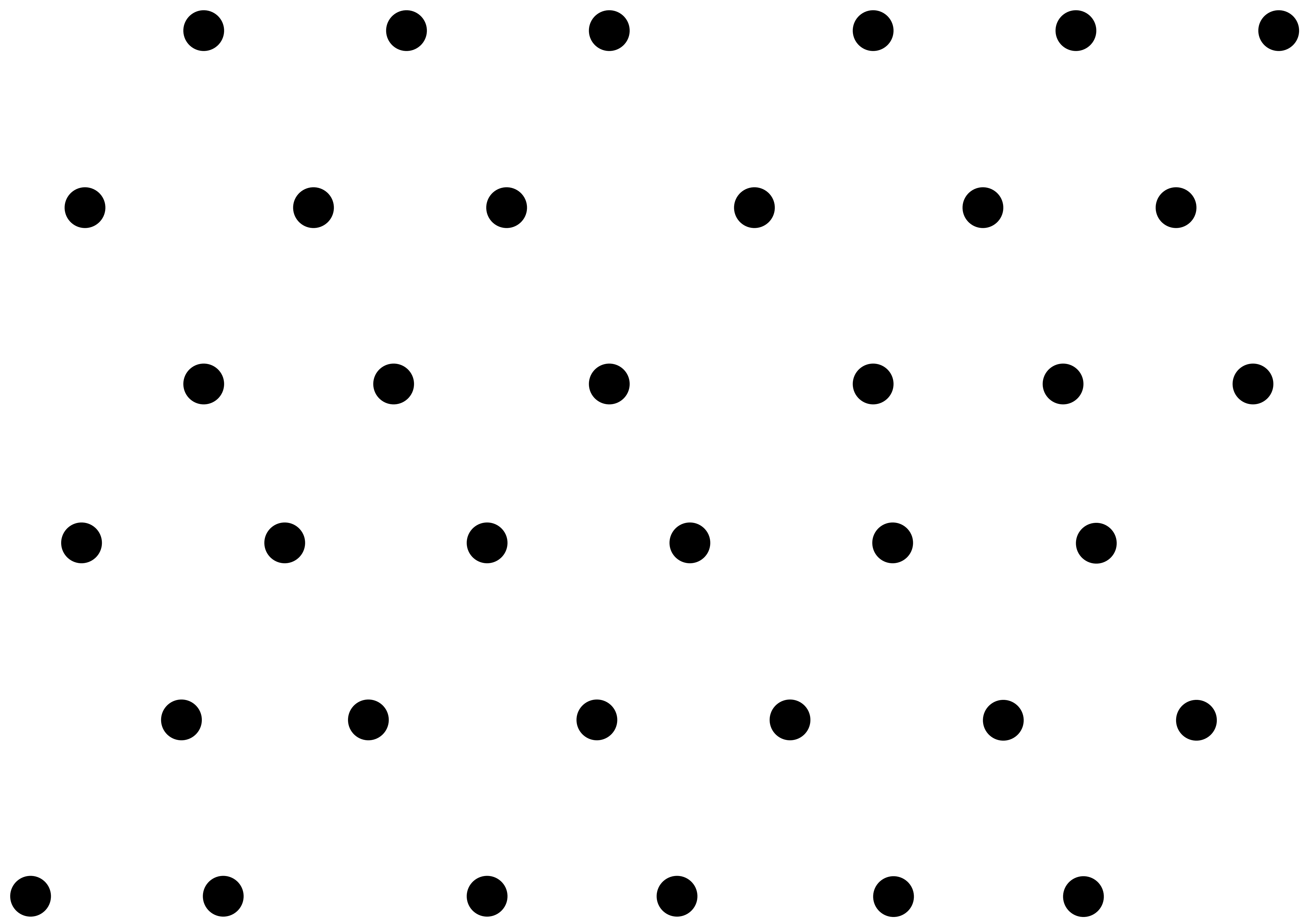
List-Decodability

$\mathscr{C} \leq \mathbb{F}_q^n$ a linear code



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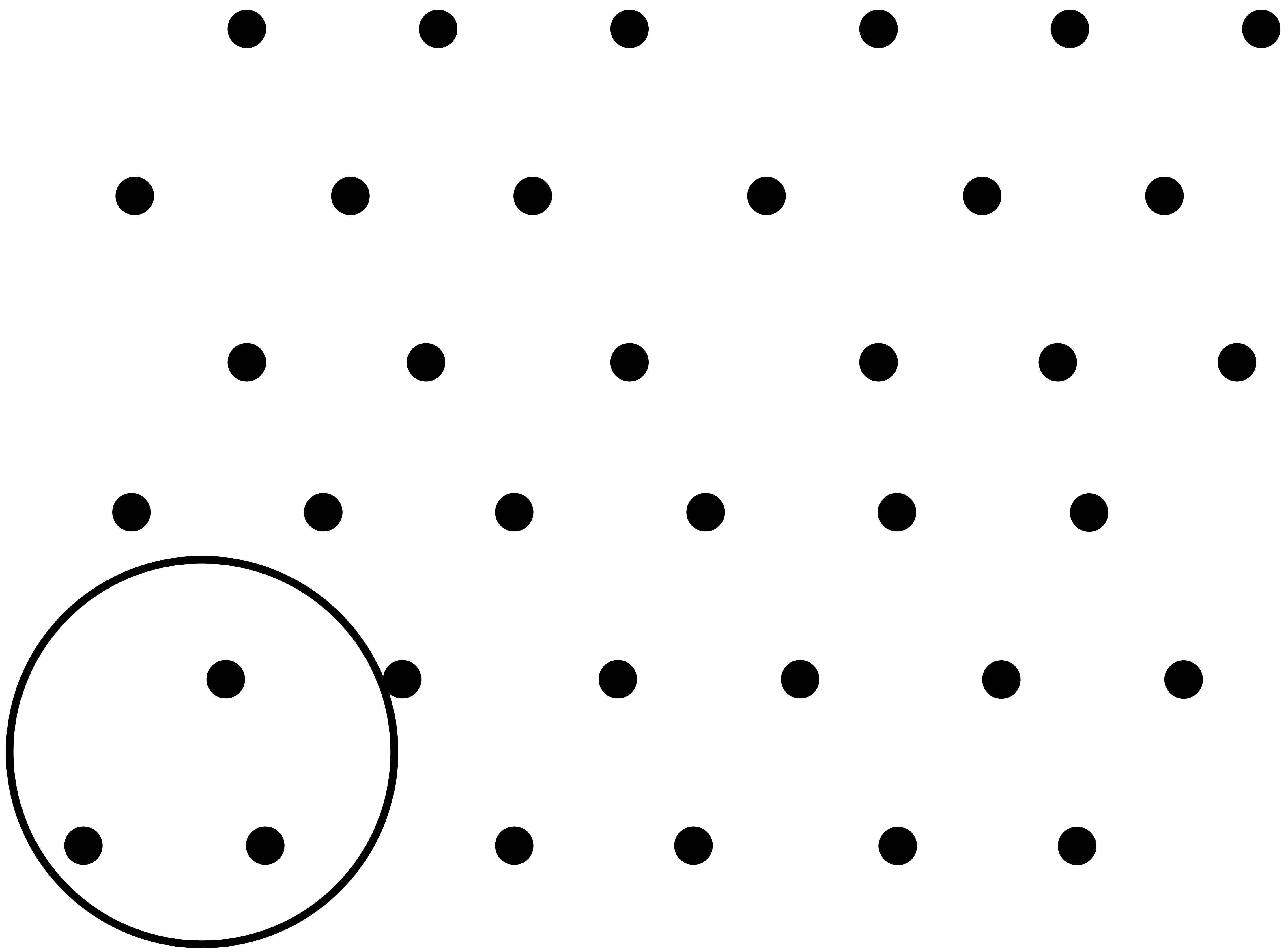

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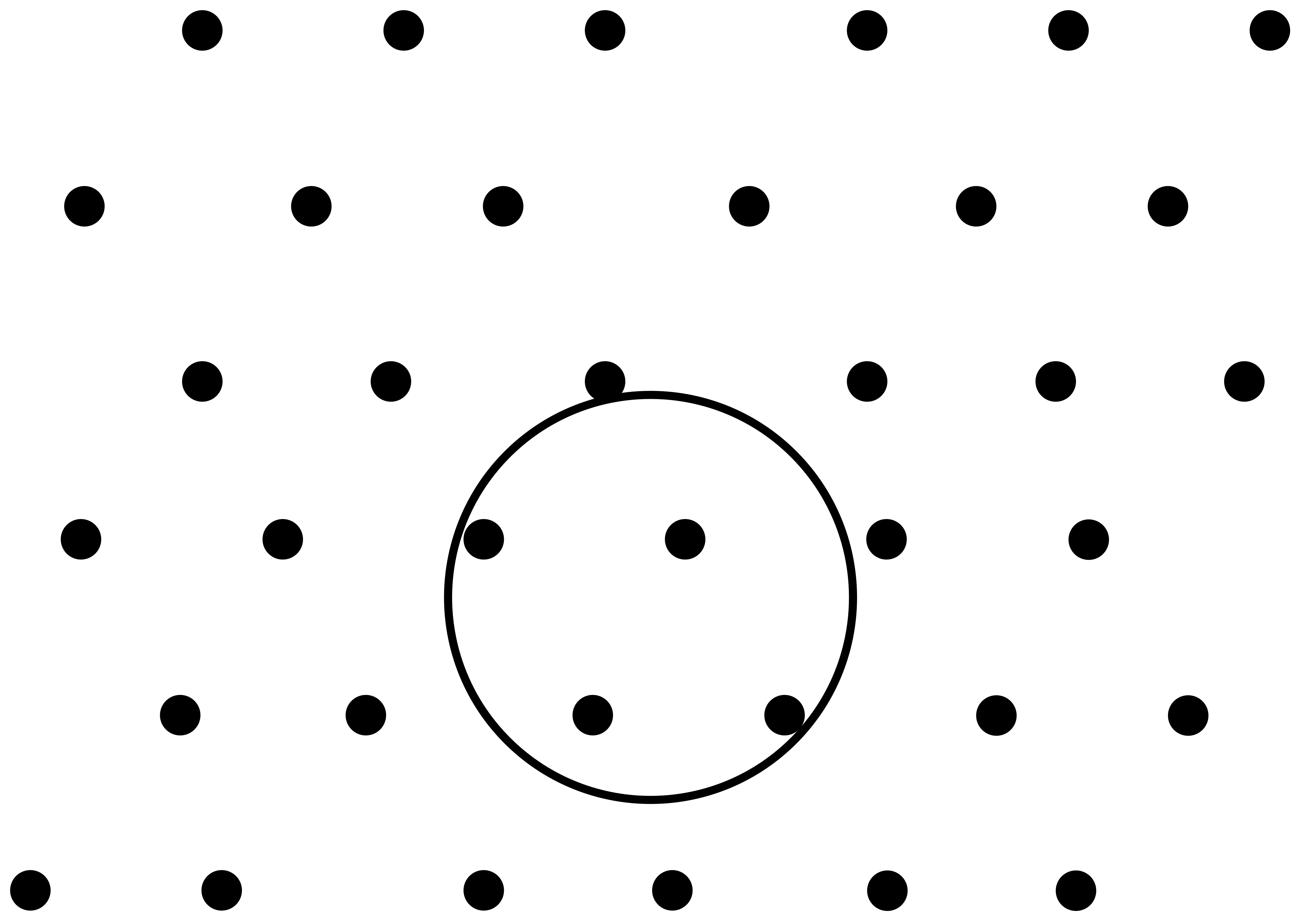

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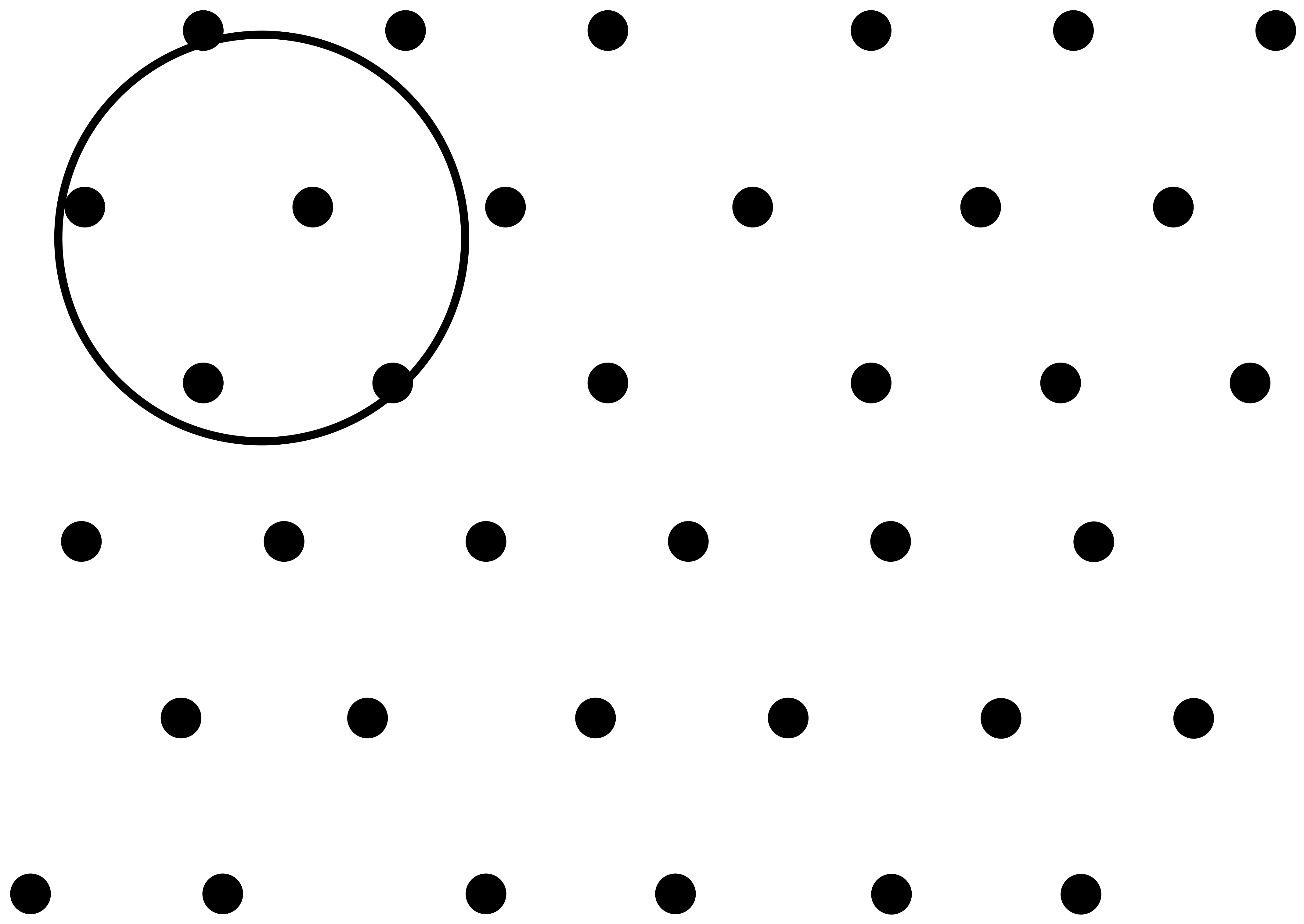

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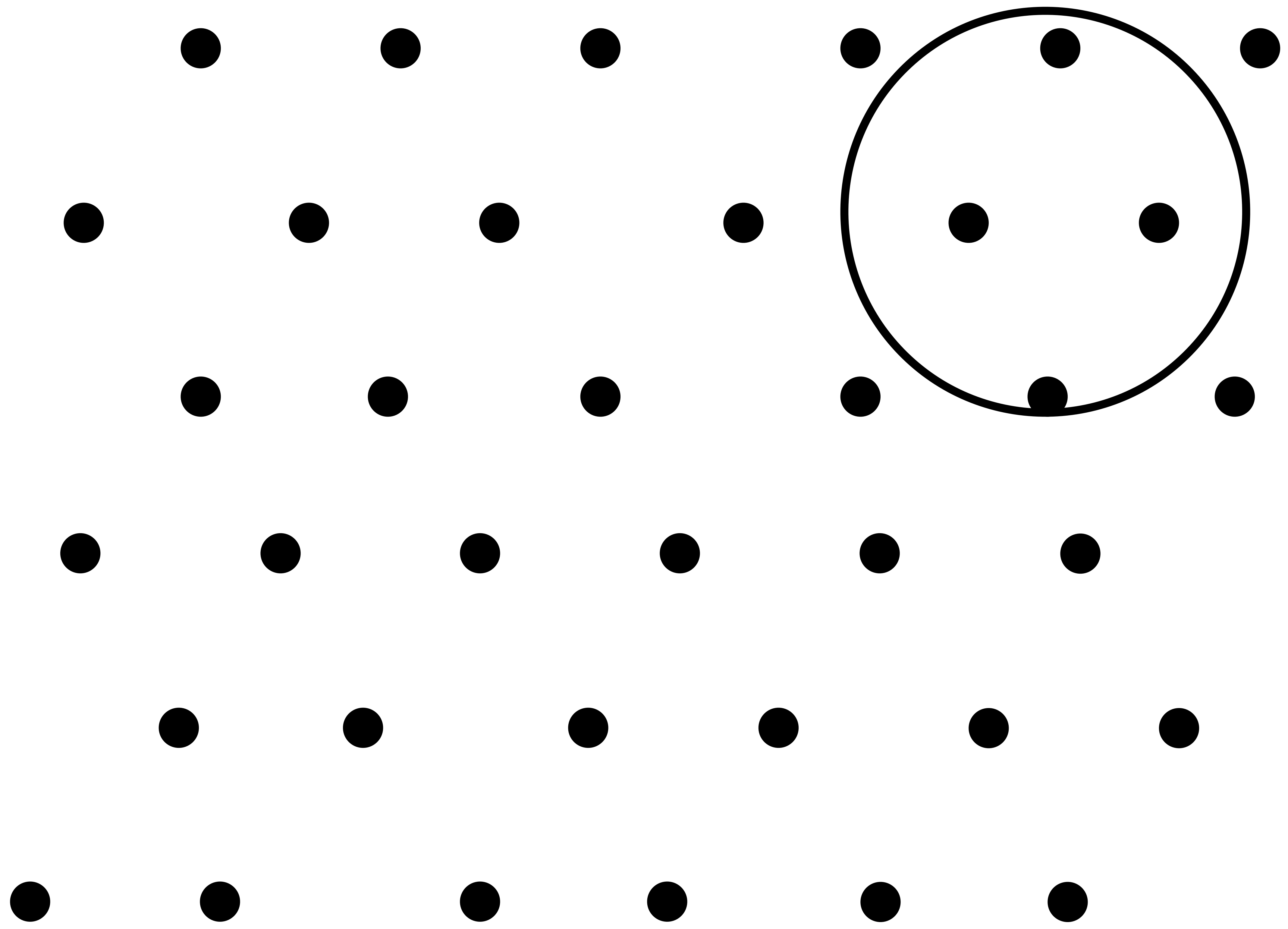

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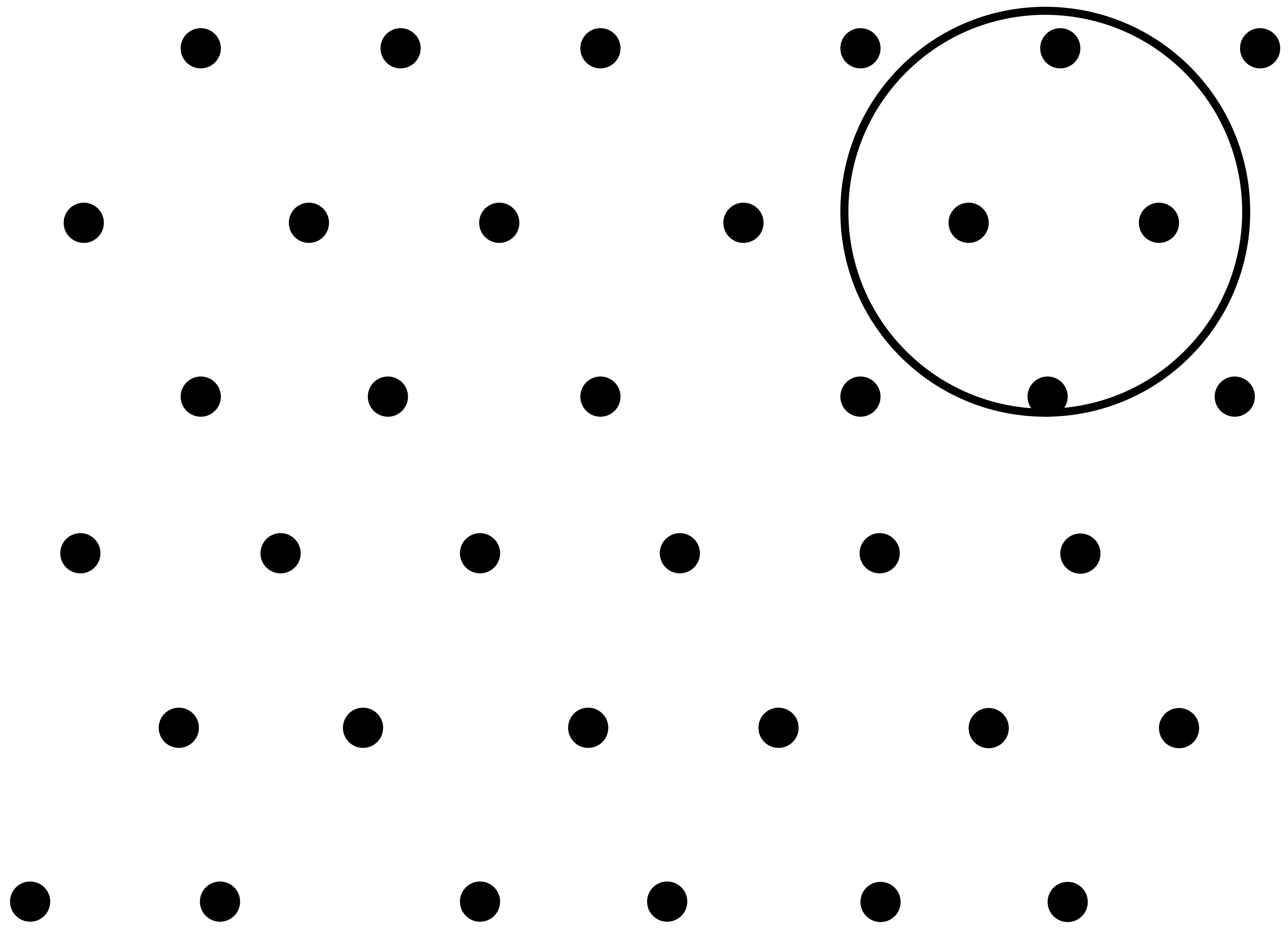

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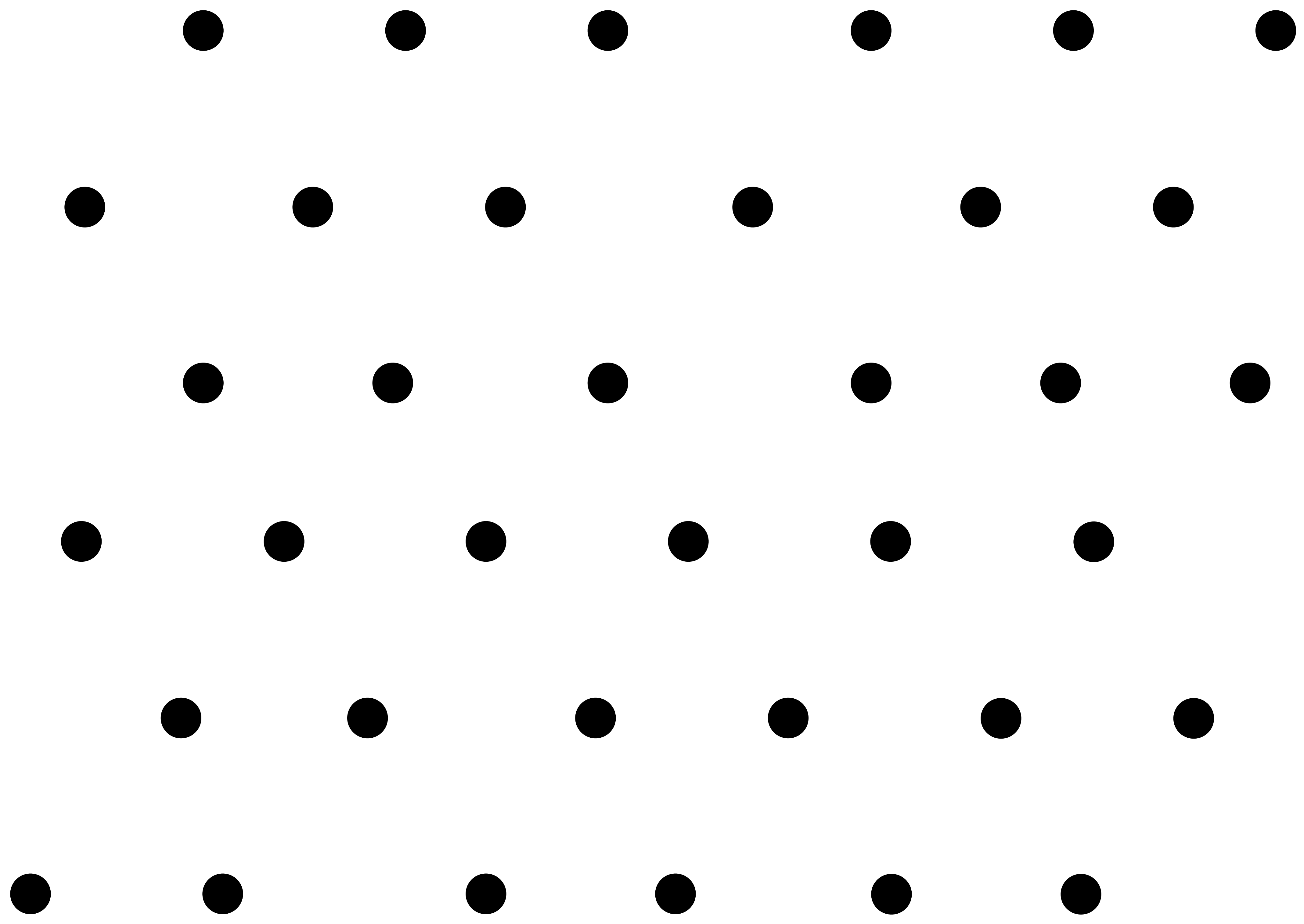
Call $\mathscr{C}$ (ρ, L)-***list-decodable*** if
$\forall z \in \mathbb{F}_q^n, |(z + B_\rho) \cap \mathscr{C}| \leq L$

For any $B \subseteq \mathbb{F}_q^n$, $z + B = \{z + x : x \in B\}$

List-Recoverability*

$C \leq \mathbb{F}_q^n$ a linear code



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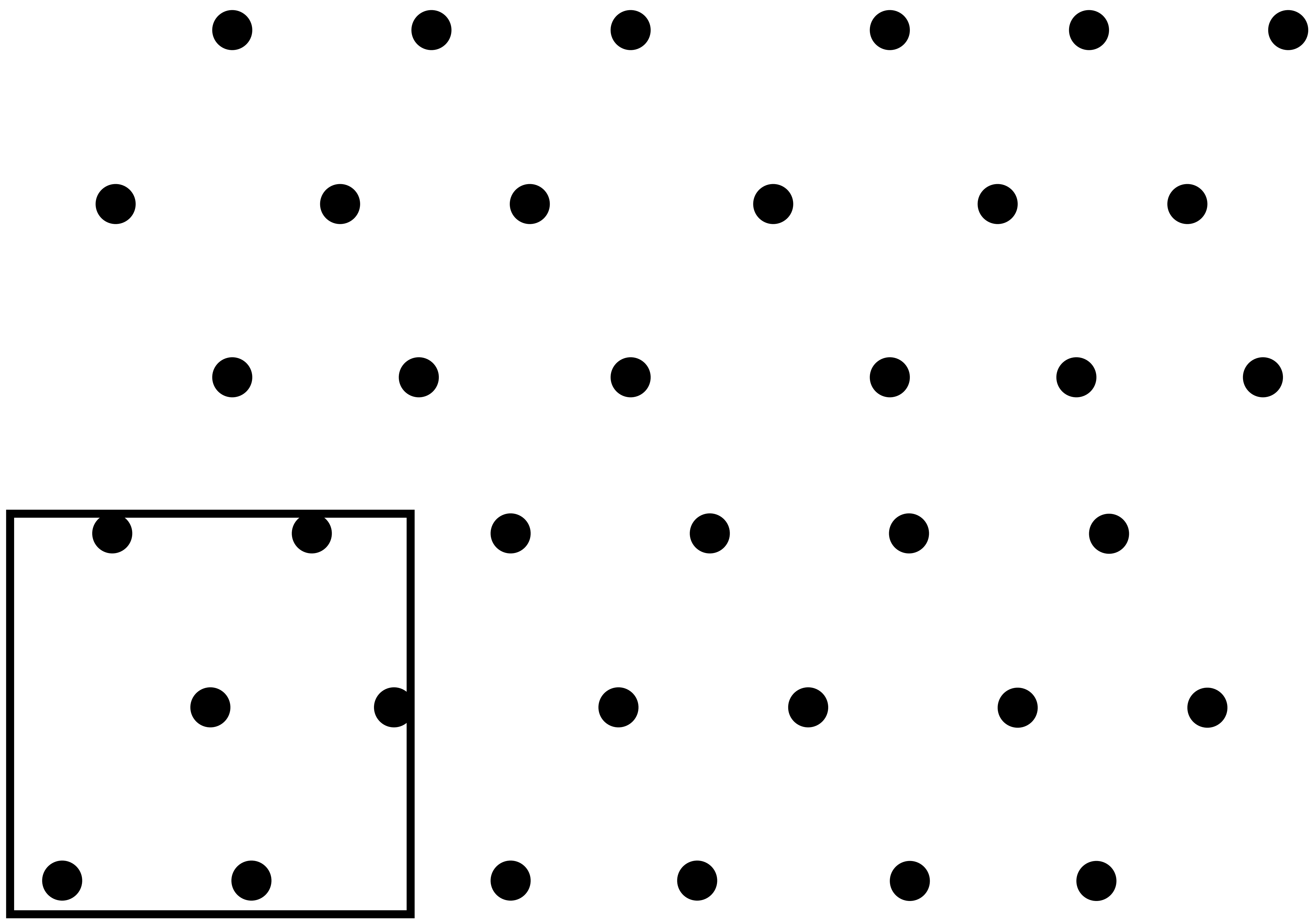

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Rectangles $S_1 \times \cdots \times S_n$

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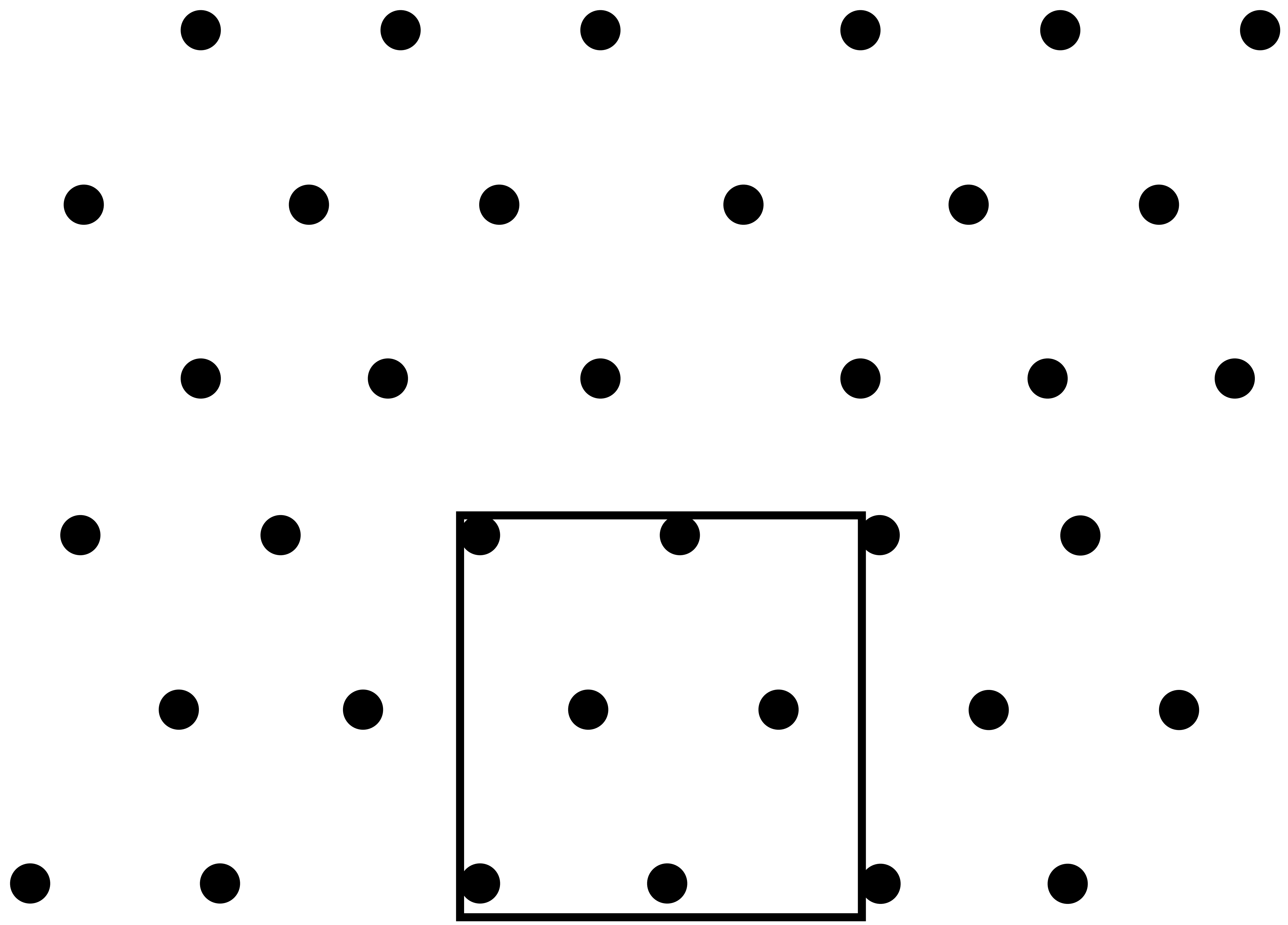

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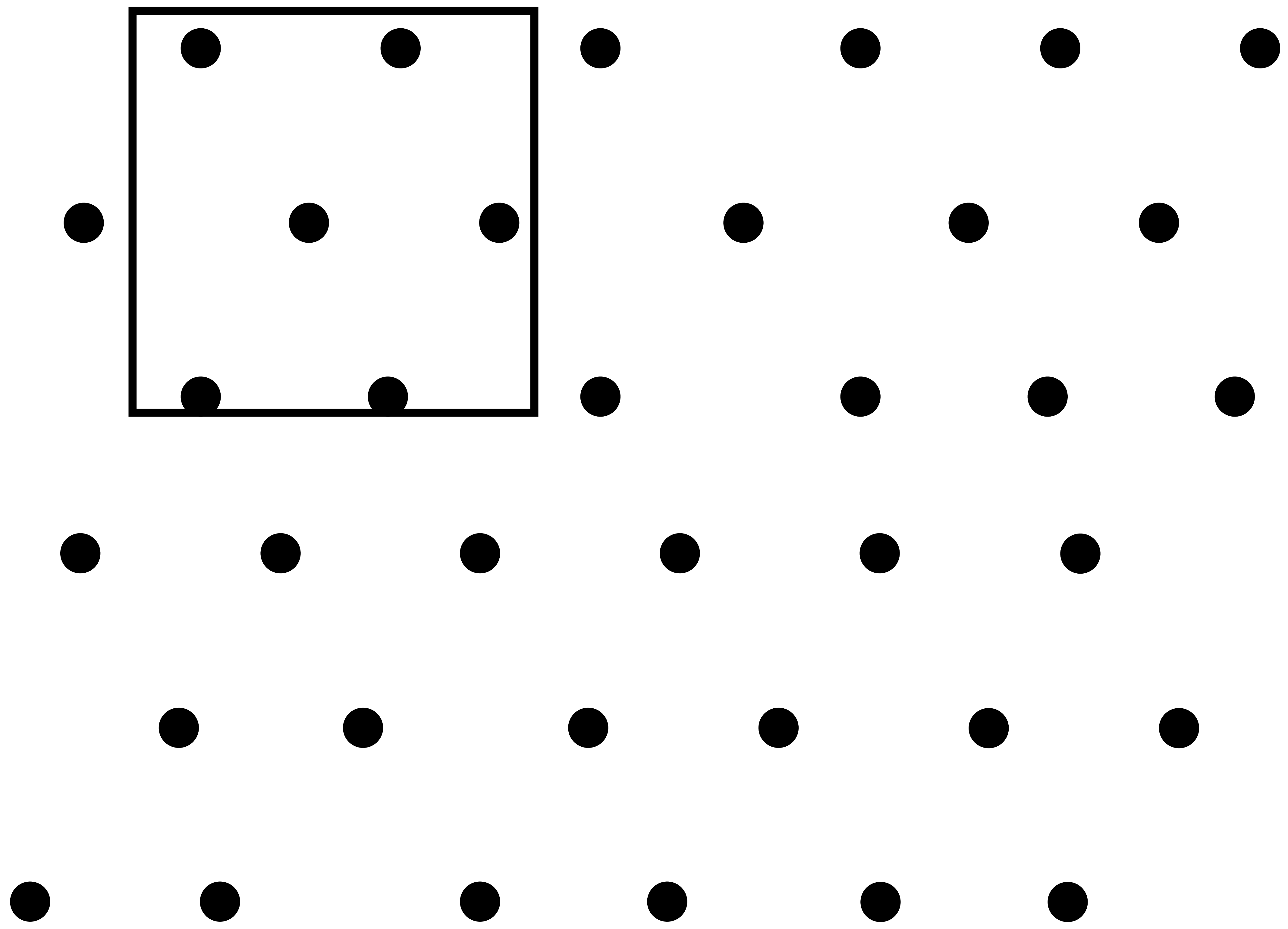

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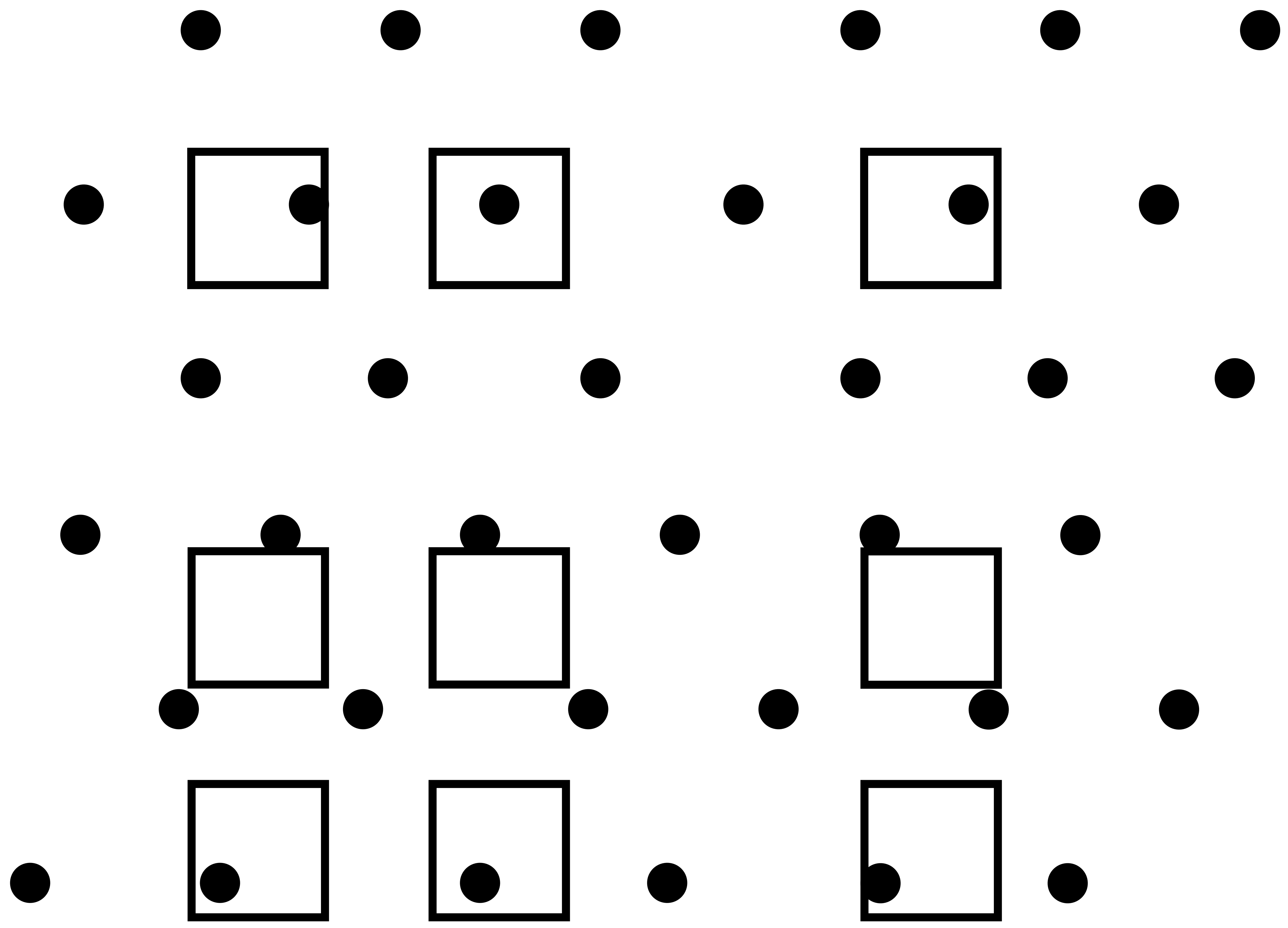

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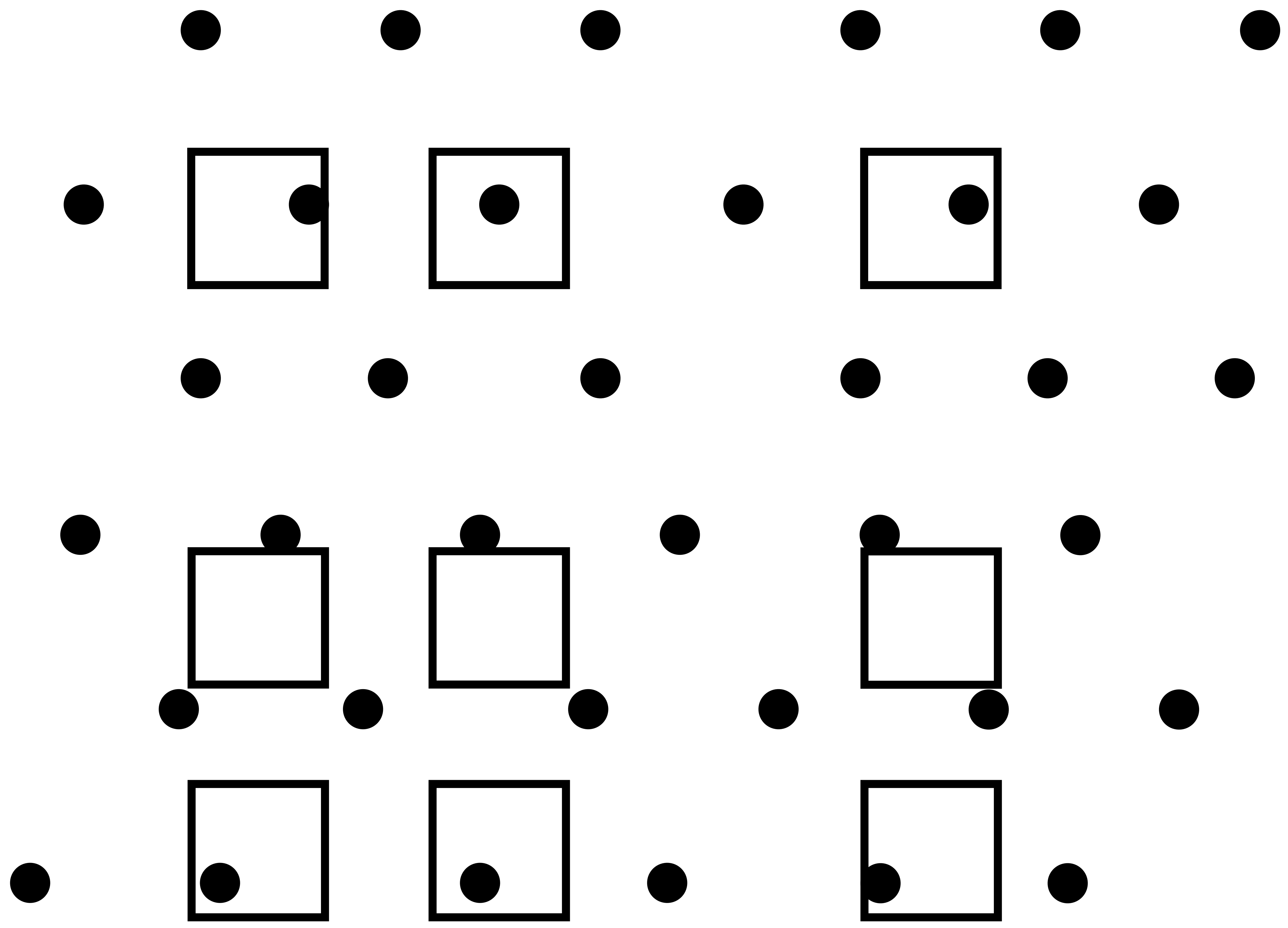

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I will boldface random variables

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Usually, study codes C at rate $\beta - \varepsilon$, so $\mathbb{E}[\,|\mathbf{C} \cap B|\,] = q^{-\varepsilon n} \ll 1$, and argue

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$$|B \cap C| \leq L \text{ for many sets } B \text{ (of same size)}$$

If code has rate $\beta + \varepsilon$, maybe we could have

$$|B \cap C| = (1 \pm o(1))\,\mathbb{E}[\,|\mathbf{C} \cap B|\,] = (1 \pm o(1))\,q^{\varepsilon n}$$

(again, for many sets B)?

Motivation?

Covering Codes

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Could hope for: $\forall z \in \mathbb{F}_q^n, \;\; |C \cap (z + B_\rho)| \approx \dfrac{|C||B_\rho|}{q^n}$

Secret-Sharing Schemes

“secret”

s

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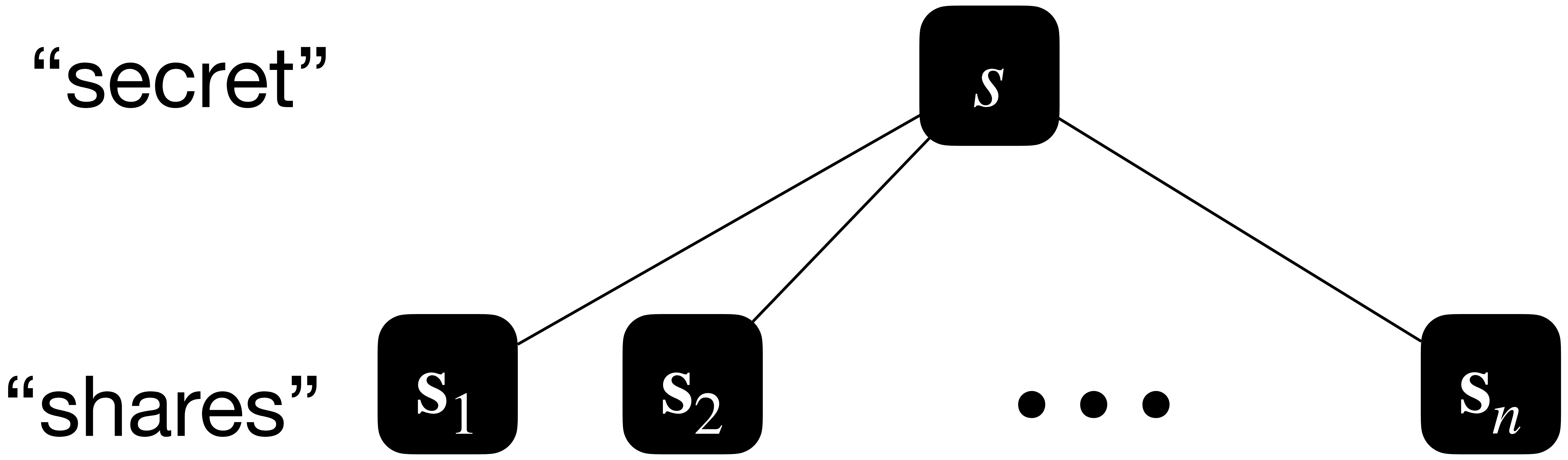

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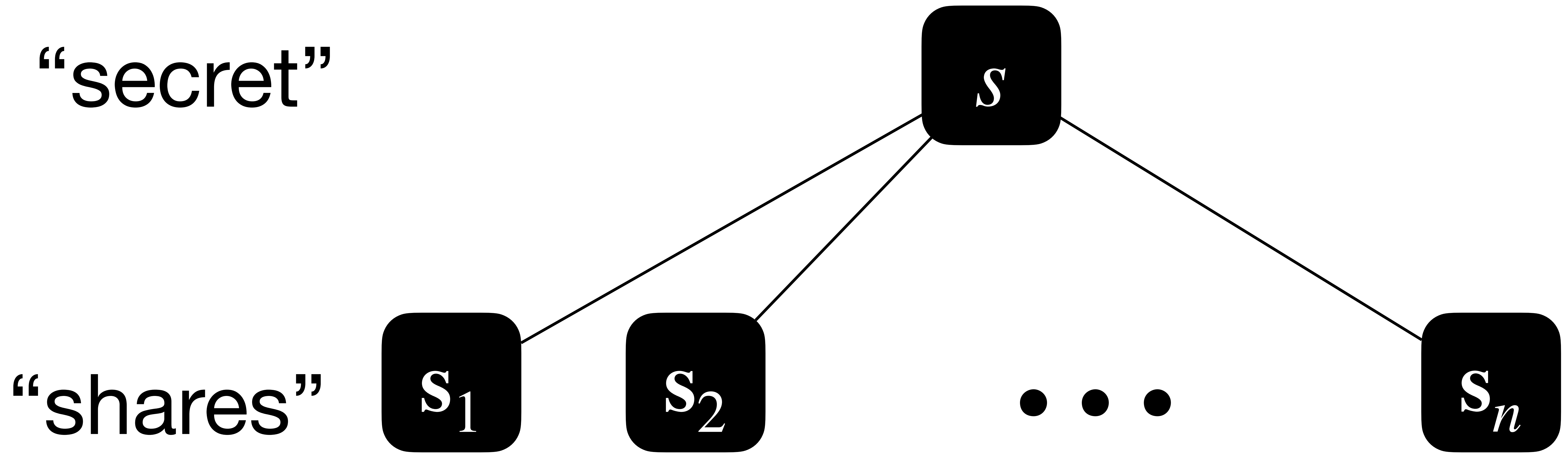

$(n, t_{\text{rec}}, t_{\text{pri}})$-secret sharing scheme:

$$\text{Recon}(\mathbf{s}_i : i \in T) = s \text{ if } |T| \geq t_{\text{rec}}$$

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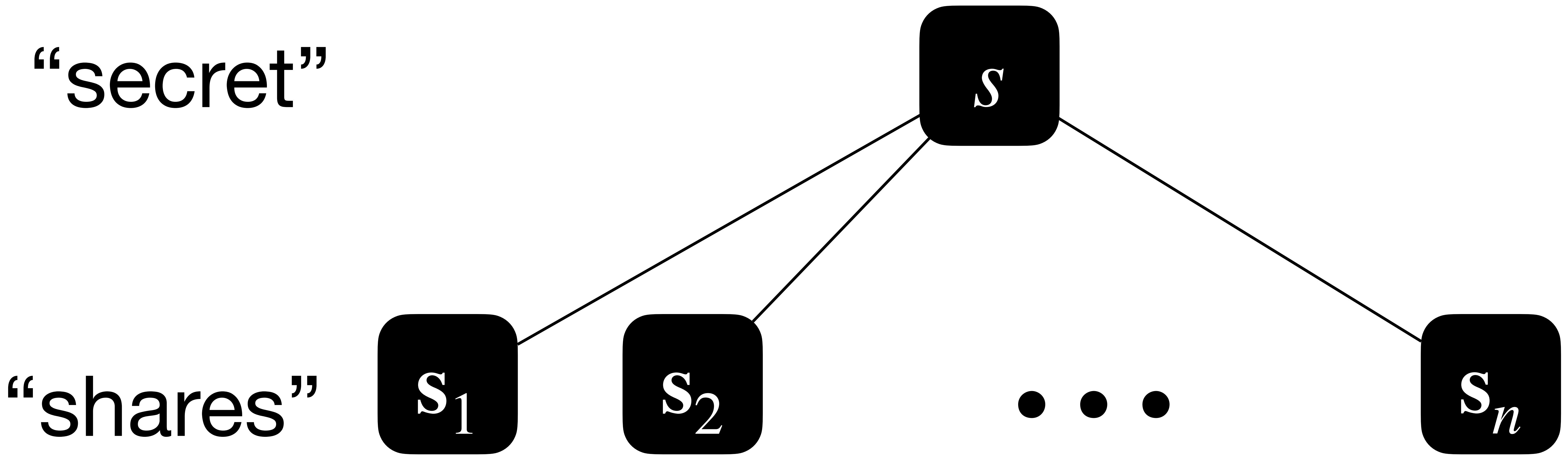

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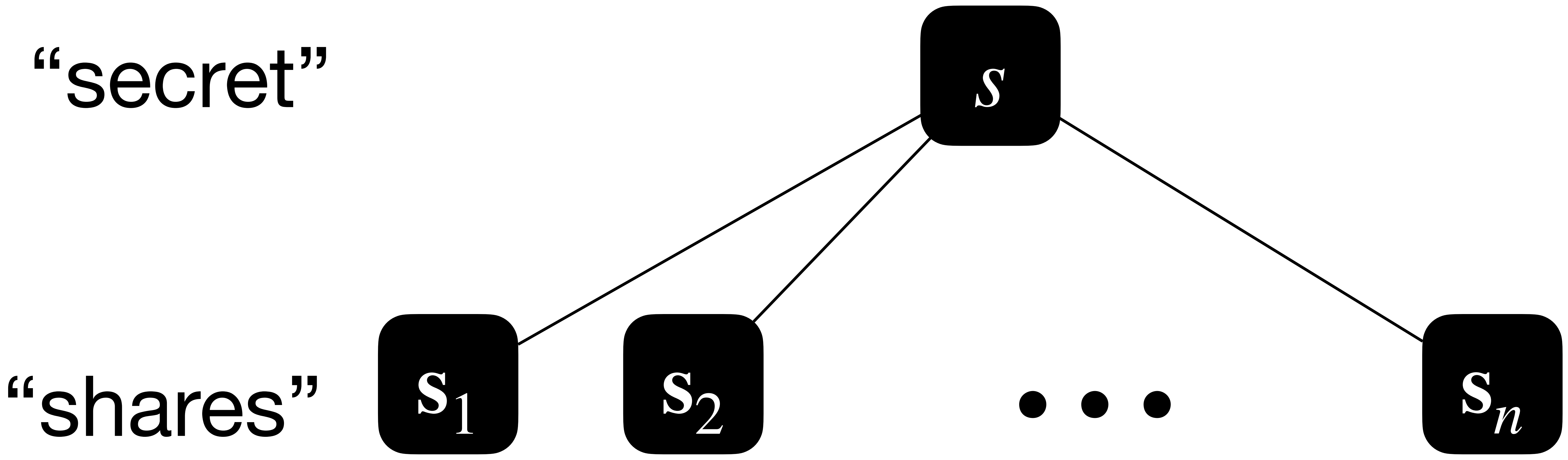

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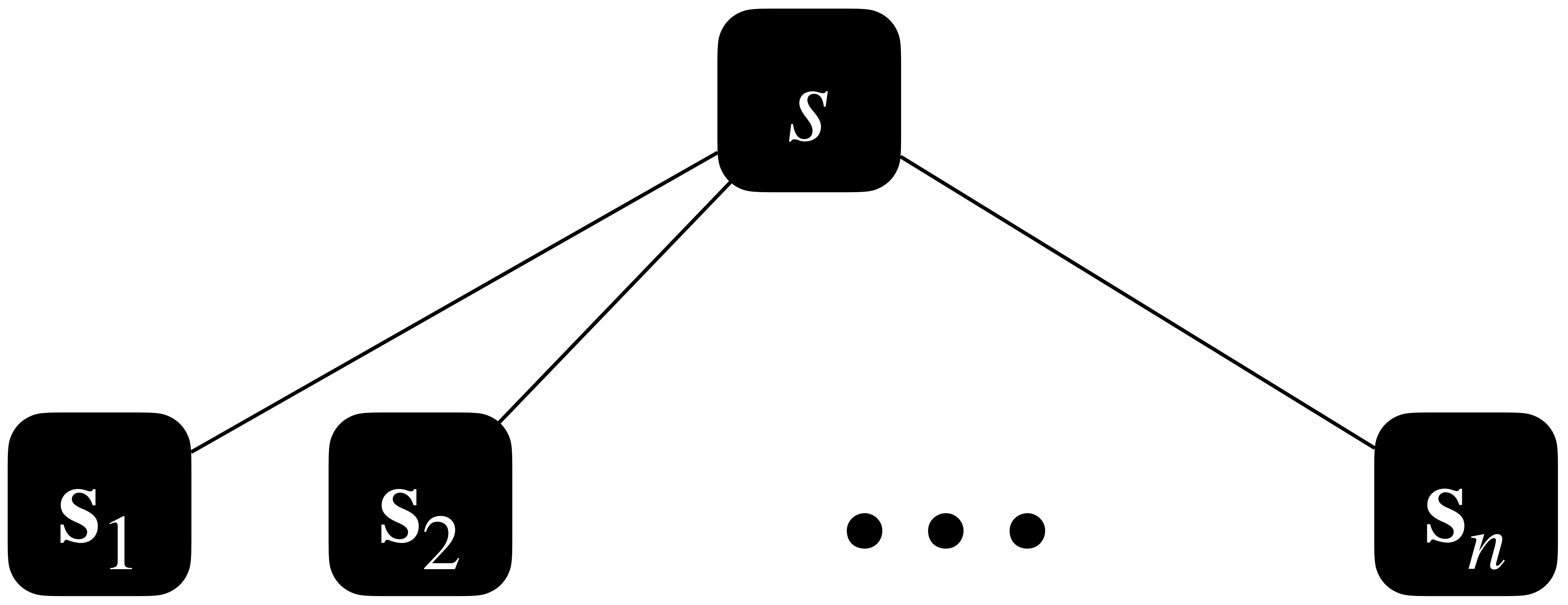
Massey's scheme: shares $(\mathbf{s}_1, \ldots, \mathbf{s}_n)$ correspond to codewords* in C, $t_{\text{rec}} = n - d(C) + 1$, $t_{\text{pri}} = d(C^{\perp}) - 1$

*technically from coset of C

(Local) Leakage Resilience



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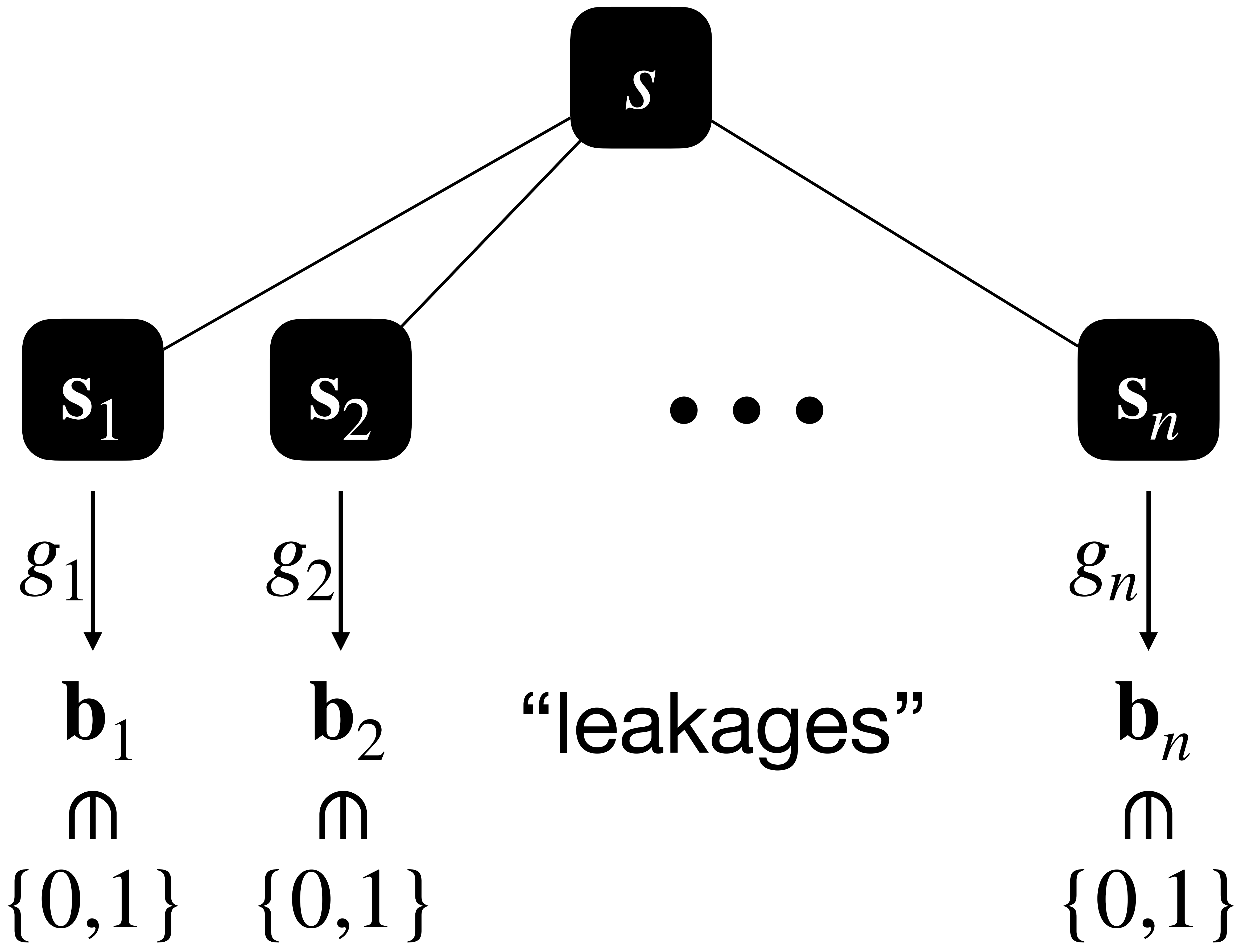

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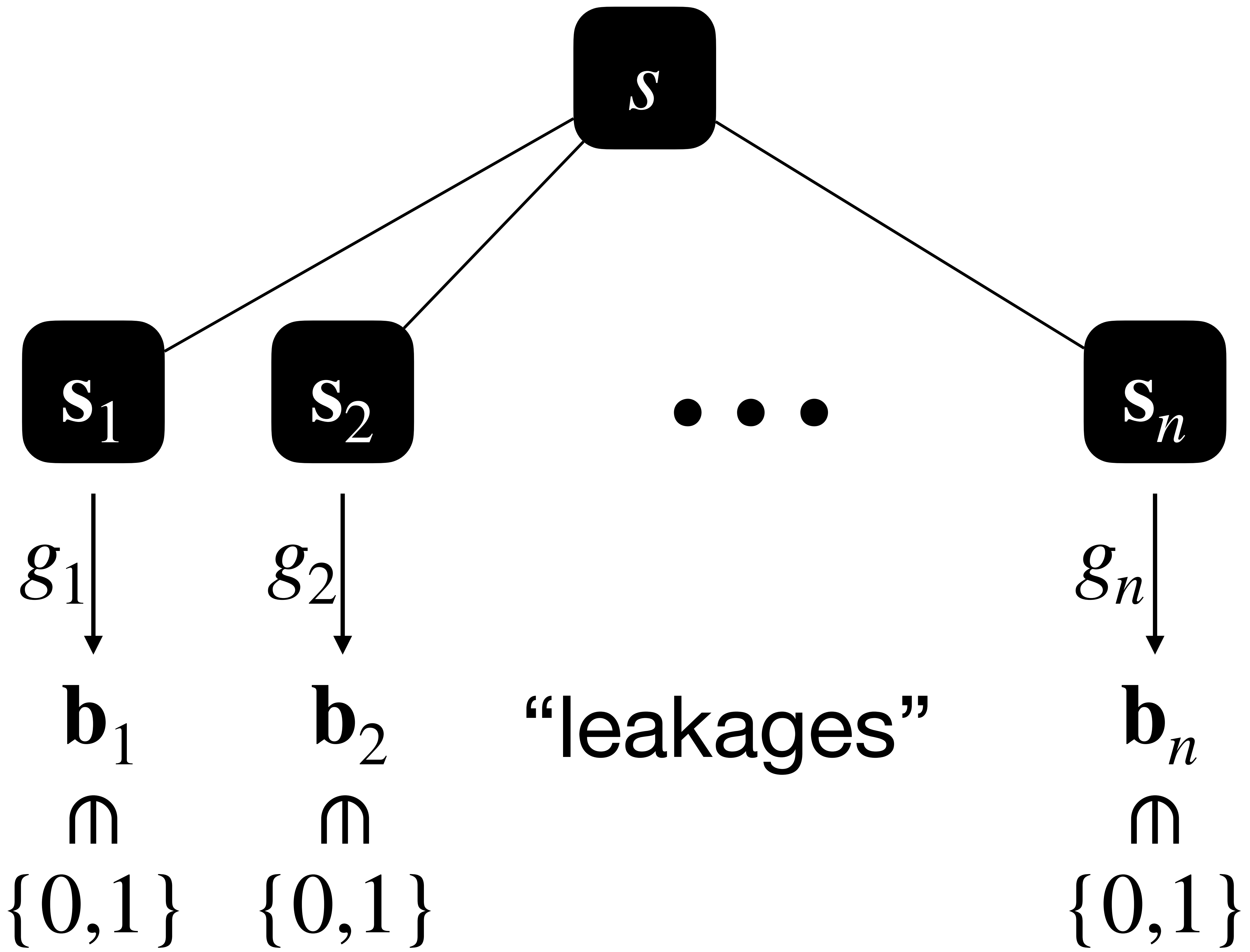

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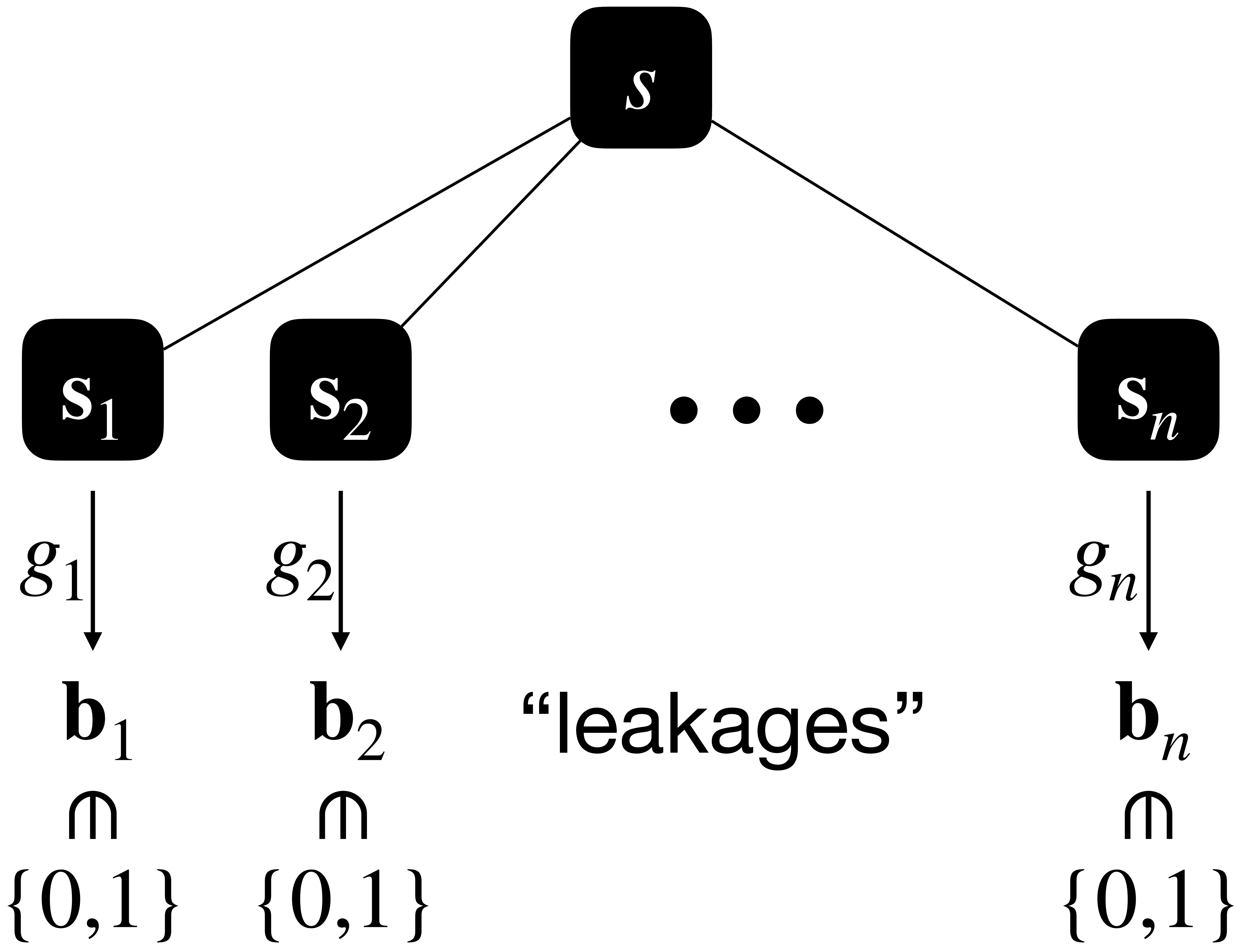

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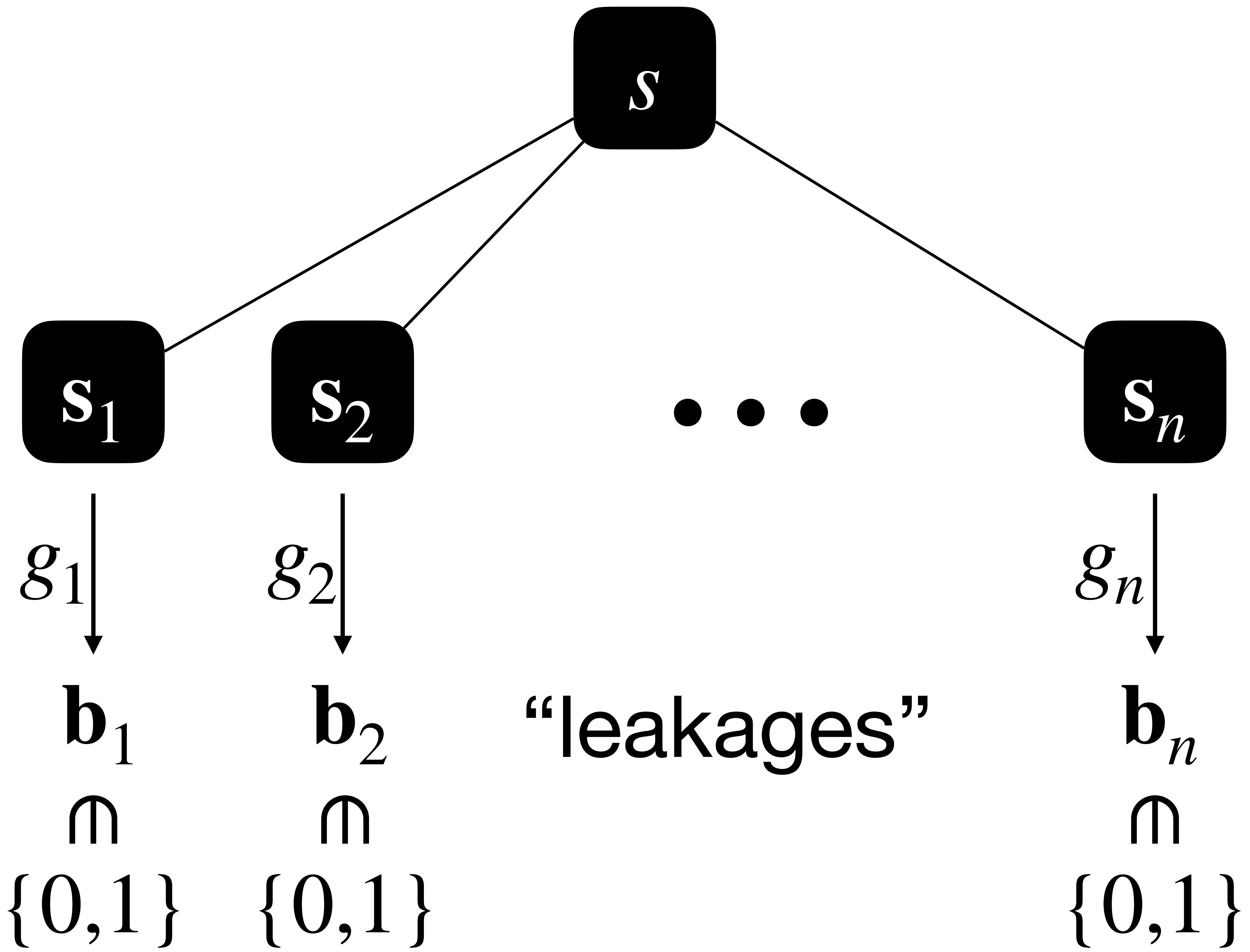

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$$|C \cap (S_1 \times \cdots \times S_n)| \approx \frac{|C|}{2^n} = \mathbb{E}[\,|\mathbf{C} \cap (S_1 \times \cdots \times S_n)|\,]$$

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Thus, line of works [MPSW21,TX22,KK23,Kas24,Ngu24] studies leakage-resilience over ***prime fields***

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Notable challenge:

Codes of rate $< 1/2$?

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Given any $S \subseteq \mathbb{F}_q^n$, wish to show:

$$\max_{z \in \mathbb{F}_q^n} \mathbb{E}\,|(z+S) \cap \mathbf{C}| \le O\left(\frac{\log n}{\log\log n}\right)$$

where $|\mathbf{C}|\,|S| = q^{n+O(1)}$

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[JSW+25,CT25,…,SW26]

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$$|\{c \in C : |\{i \in [n] : c_i \notin S_i\}| \leq \rho n\}| \approx q^{\Omega(n)}$$

find one codeword in above set

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Our case: natural (?) coding-theoretic variant

Our Results

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Applies to Hamming balls!
Generalizes covering-radius result

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Theorem 2: Pseudorandom Sets

Let $\mathscr{B}$ be a family of sets with $|\mathscr{B}| \leq q^{o(n^2)}$ where each set B is "pseudorandom,"* let $\beta = 1 - \frac{1}{n}\log_q \min\{|B| : B \in \mathscr{B}\}$, and let $\mathbf{C}$ be RLC of rate $\beta + \varepsilon$. Then, with prob. $1 - q^{-\Omega(n)}$,

$$\forall B \in \mathscr{B},\ \ |\mathbf{C} \cap B| = \left(1 \pm q^{\Omega(-\varepsilon n)}\right) \cdot \frac{|\mathbf{C}||B|}{q^n}$$

Applies to Rectangles over *prime fields*,**
i.e., list-recovery

*$\hat{1}_B(y)$ decays exponentially in $\mathrm{wt}(y)$

**assuming q, ℓ not too big

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Corollary: Secret-sharing scheme associated to RLC $\mathbf{C}$ of rate ε is whp leakage-resilient against *balanced* leakages, assuming q is not too large

More generally: Theorems work w/ non-negative functions, i.e., *weighted* sets

Proof Ideas (for Theorem 1)

Relative Deviation

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For set B and linear code C, define ***relative deviation***

$$\delta_{C,B} := \frac{\Pr_{\mathbf{c}\sim C}[\mathbf{c} \in B]}{\Pr_{\mathbf{x}\sim \mathbb{F}_q^n}[\mathbf{x} \in B]} - 1 = \frac{q^n \cdot |C \cap B|}{|C||B|} - 1$$

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Goal: if $\mathbf{C}$ is RLC of rate $\beta + \varepsilon$, $|\delta_{\mathbf{C},B_z}| \leq q^{-\Omega(\varepsilon n)}$ for all z whp, where $B_z := z + B$

Iterative RLC Construction

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Sample i.i.d. uniform $\mathbf{u}_1, \ldots, \mathbf{u}_k$ for $k = (\beta + \varepsilon)n$, and define

$$C_0 := \{0\}, \mathbf{C}_i := \mathbf{C}_{i-1} + \langle \mathbf{u}_i \rangle$$

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[Blinovsky'87] analyzed procedure for covering radius

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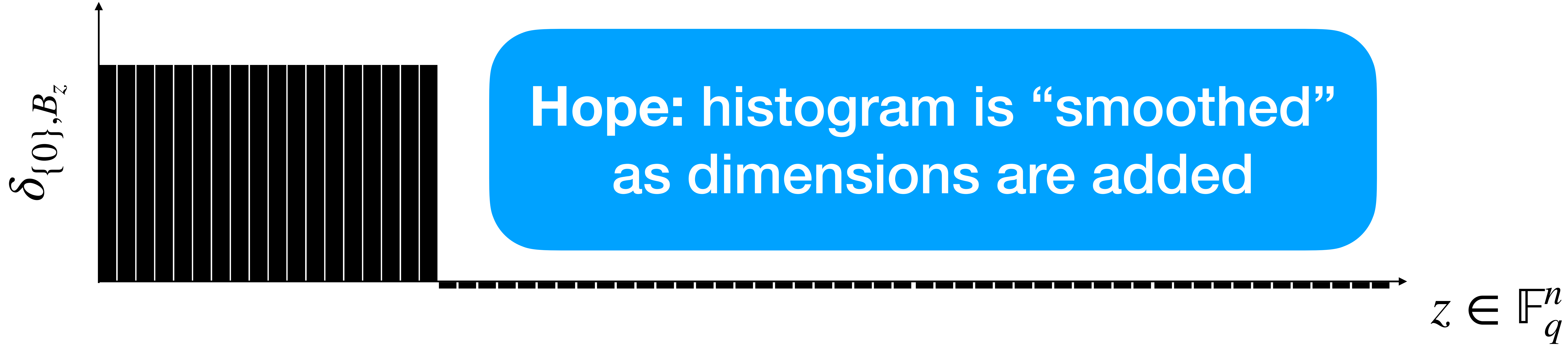

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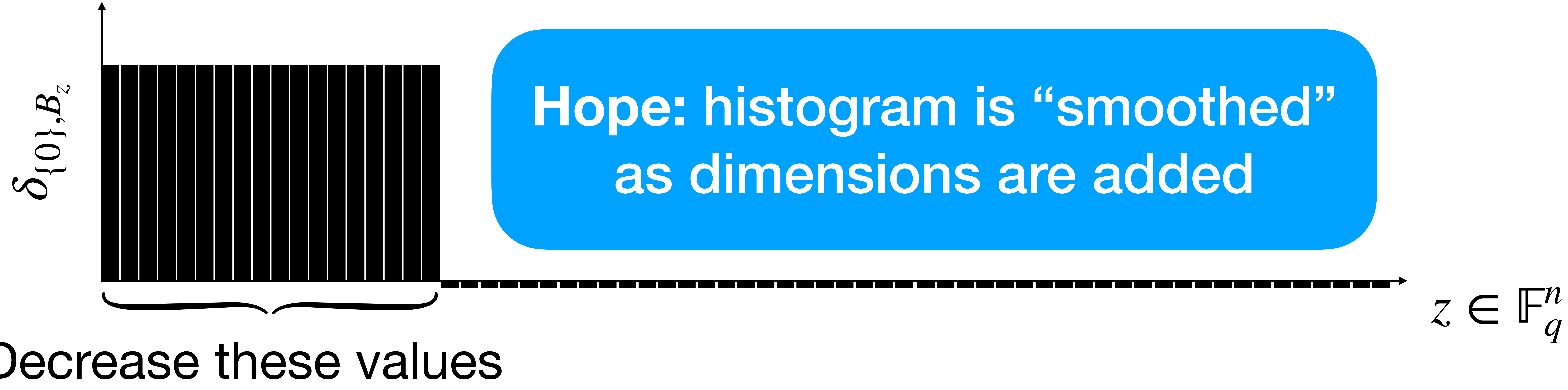

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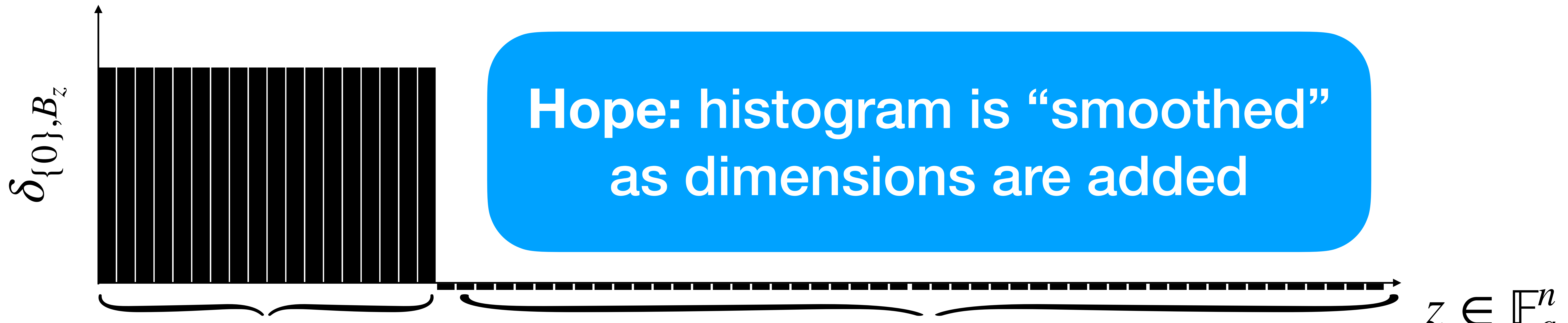

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Decrease these values

Increase these values

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$$\mathbb{E}\left[\delta_{\mathbf{C}',B}\right] = \frac{1}{q} \sum_{\lambda \in \mathbb{F}_q} \mathbb{E}\left[\delta_{C+\lambda\mathbf{u},B}\right] = \frac{\delta_{C,B}}{q}.$$

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Just need "most" steps good; $(1 - \varepsilon/2)$ fraction, say

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$\lambda > 0$ small slack

condition on C_{i-1}

Call step i good for z if

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Via 2nd moment method: can bound

$$\mathbb{P}\left[|\delta_{\mathbf{C}_i,B_z}| \le \left(\frac{1}{q}+\lambda\right)|\delta_{C_{i-1},B_z}|\right] \le q^{-\Omega(\varepsilon n)},$$

and therefore, whp, $|\delta_{\mathbf{C}_k,B_z}| \le q^{-\Omega(\varepsilon n)}$

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$$\mathrm{Var}(\delta_{\mathbf{C}',B_z}) = \sum_{x,y \in B_z} \left(\mathbb{P}[x, y \in \mathbf{C}'] - \mathbb{P}[x \in \mathbf{C}'] \cdot \mathbb{P}[y \in \mathbf{C}']\right)$$

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Thus, we can show that whp, the construction sequence $C_0 = \{0\}, \mathbf{C}_1, \ldots, \mathbf{C}_k$ is ***smooth*** for $B - B$:

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$$\Lambda = \sqrt{\Gamma}$$

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Denominator: use second condition

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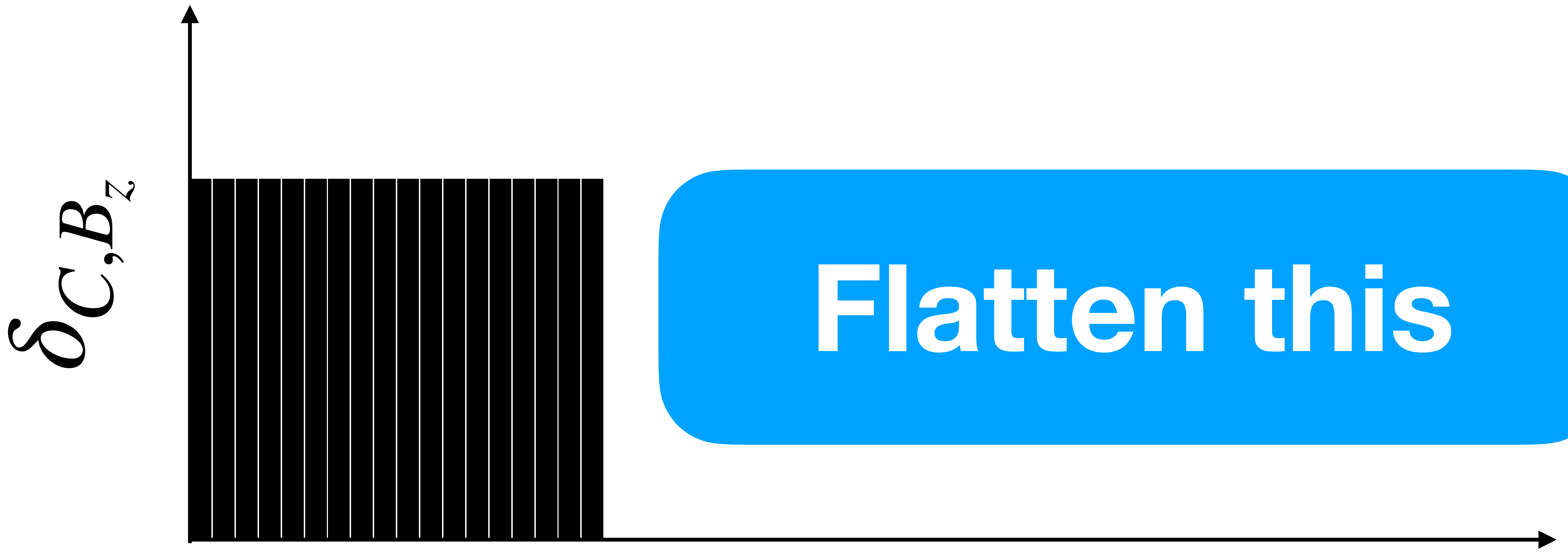
Thus,

$$\mathbb{P}[\text{enough bad steps}] \leq (q^{-\Omega(\varepsilon n)})^{\Omega(\varepsilon n)} = q^{-\Omega(\varepsilon^2 n^2)}.$$

Summary

Goal



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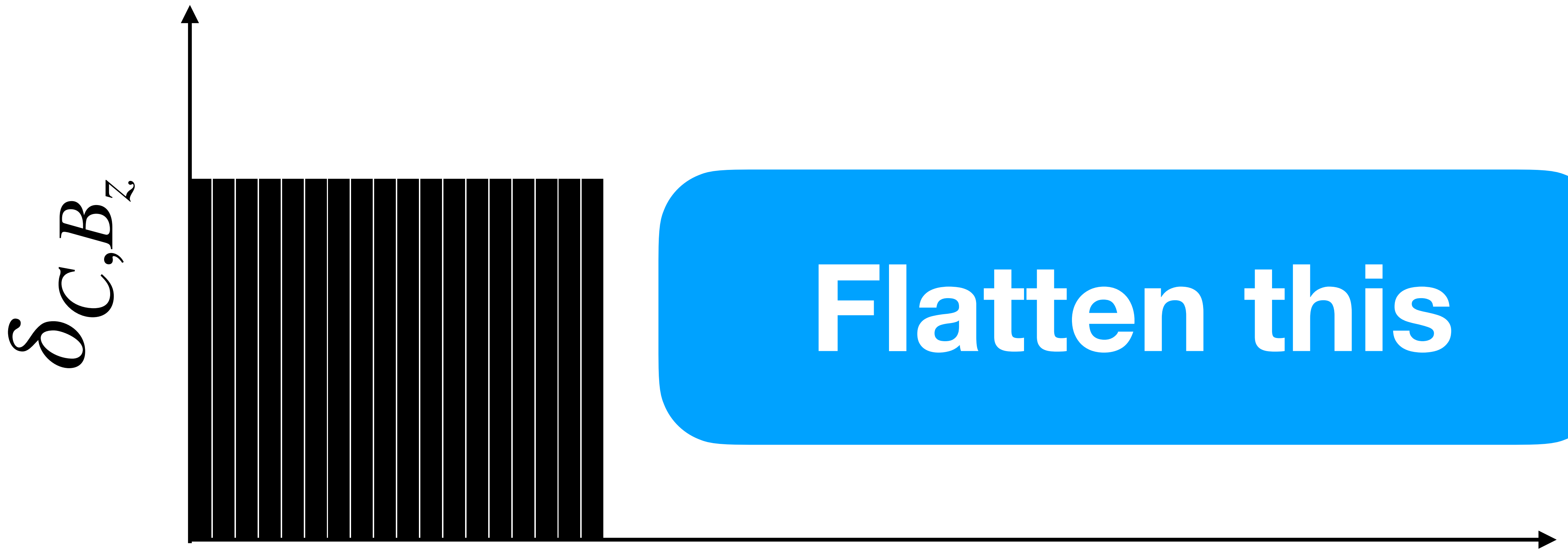

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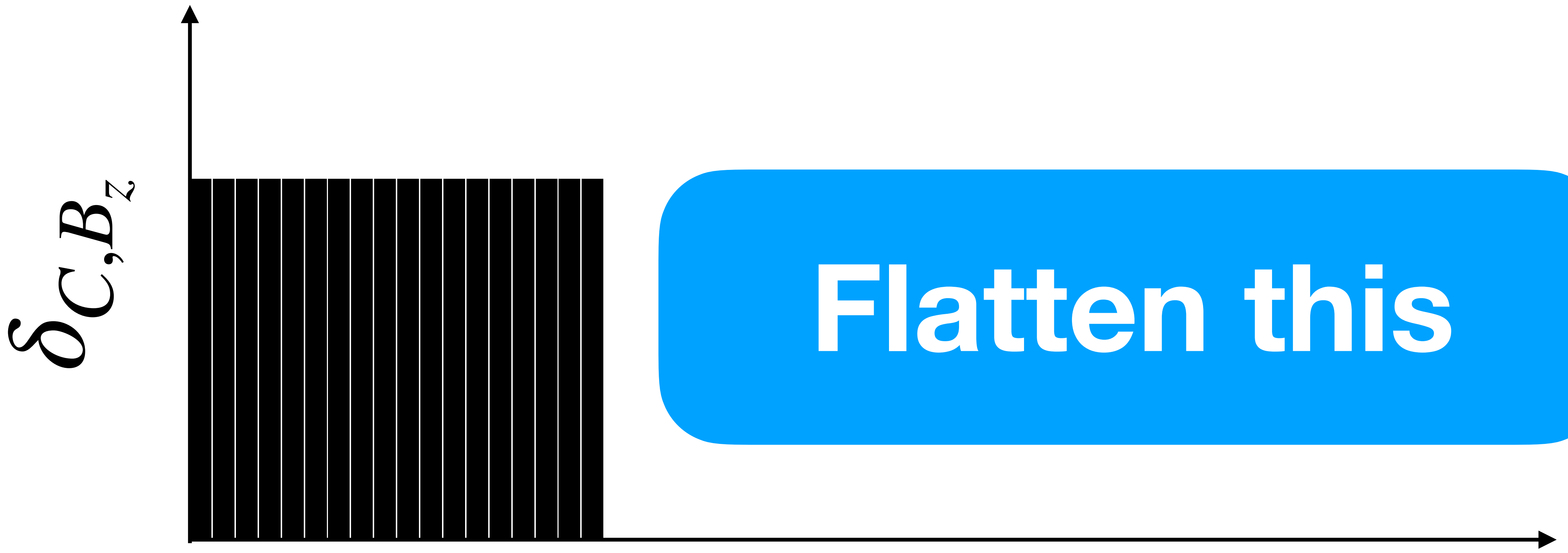

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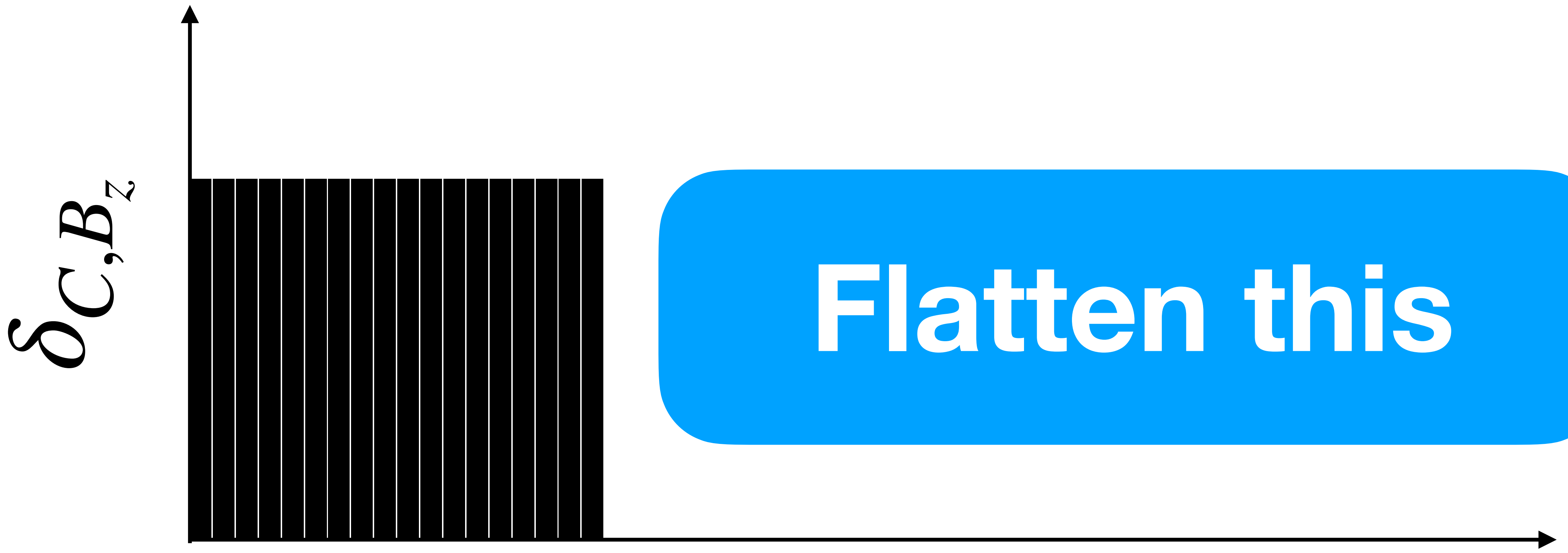

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Transfer Smoothness

$\mathbb{P}[B-B$ smooth but $\exists z$ s.t. B_z not smooth$] \leq q^n \cdot q^{-\Omega(n^2)}$

A Word About Theorem 2

Theorem 2: Pseudorandom Sets

Let $\mathscr{B}$ be a family of sets with $|\mathscr{B}| \leq q^{o(n^2)}$ where each set B is "pseudorandom,"* let $\beta = 1 - \frac{1}{n}\log_q \min\{|B| : B \in \mathscr{B}\}$, and let $\mathbf{C}$ be RLC of rate $\beta + \varepsilon$. Then, with prob. $1 - q^{-\Omega(n)}$,

$$\forall B \in \mathscr{B}, \quad |\mathbf{C} \cap B| = \left(1 \pm q^{\Omega(-\varepsilon n)}\right) \cdot \frac{|\mathbf{C}||B|}{q^n}$$

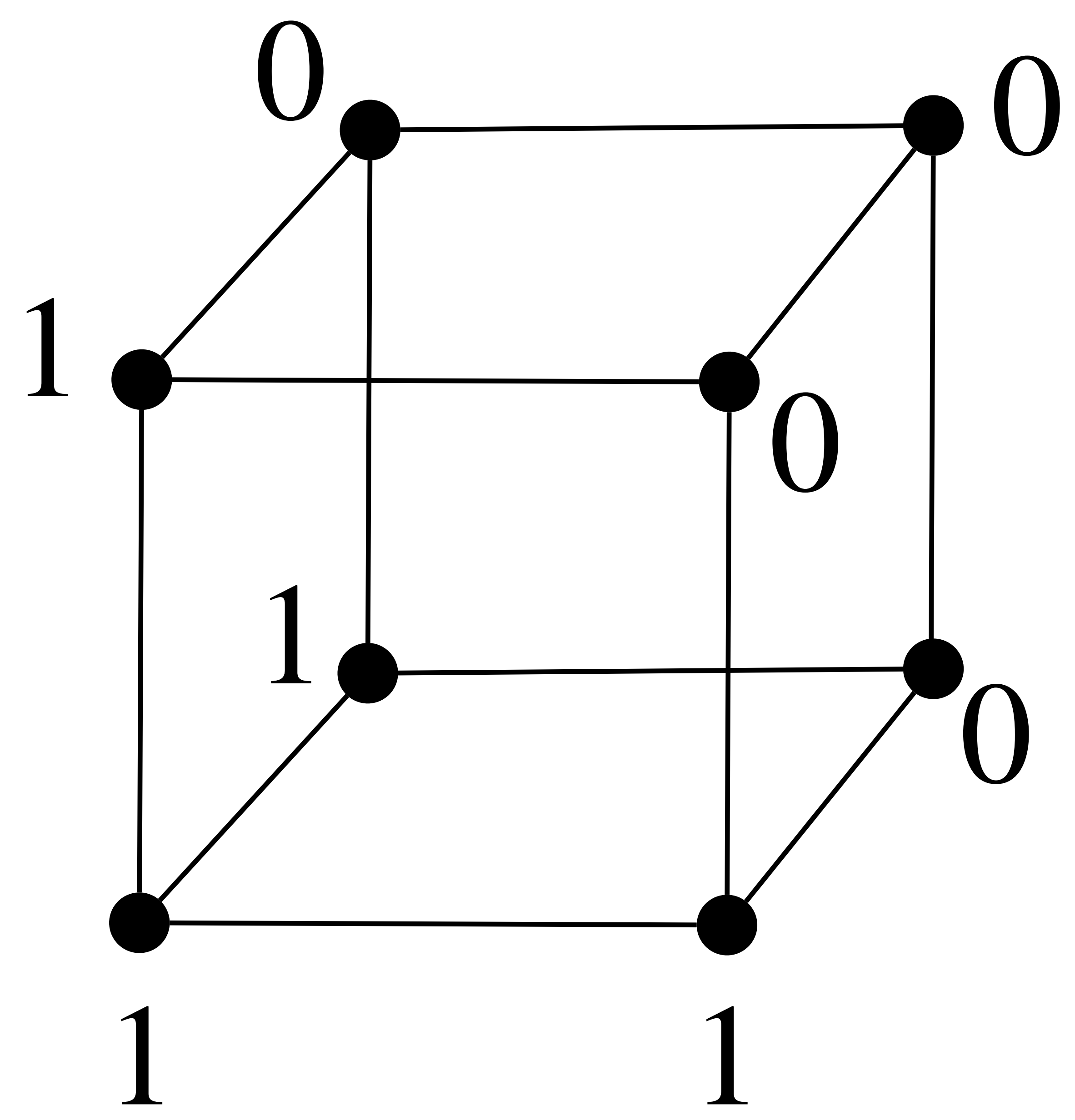
Applies to Rectangles over *prime fields*,**
i.e., list-recovery

*$\hat{1}_B(y)$ decays exponentially in $\mathrm{wt}(y)$

**assuming q, ℓ not too big

$B - B =$ **more uniform**

$B_{1/3} = \{000, 001, 010, 100\}$



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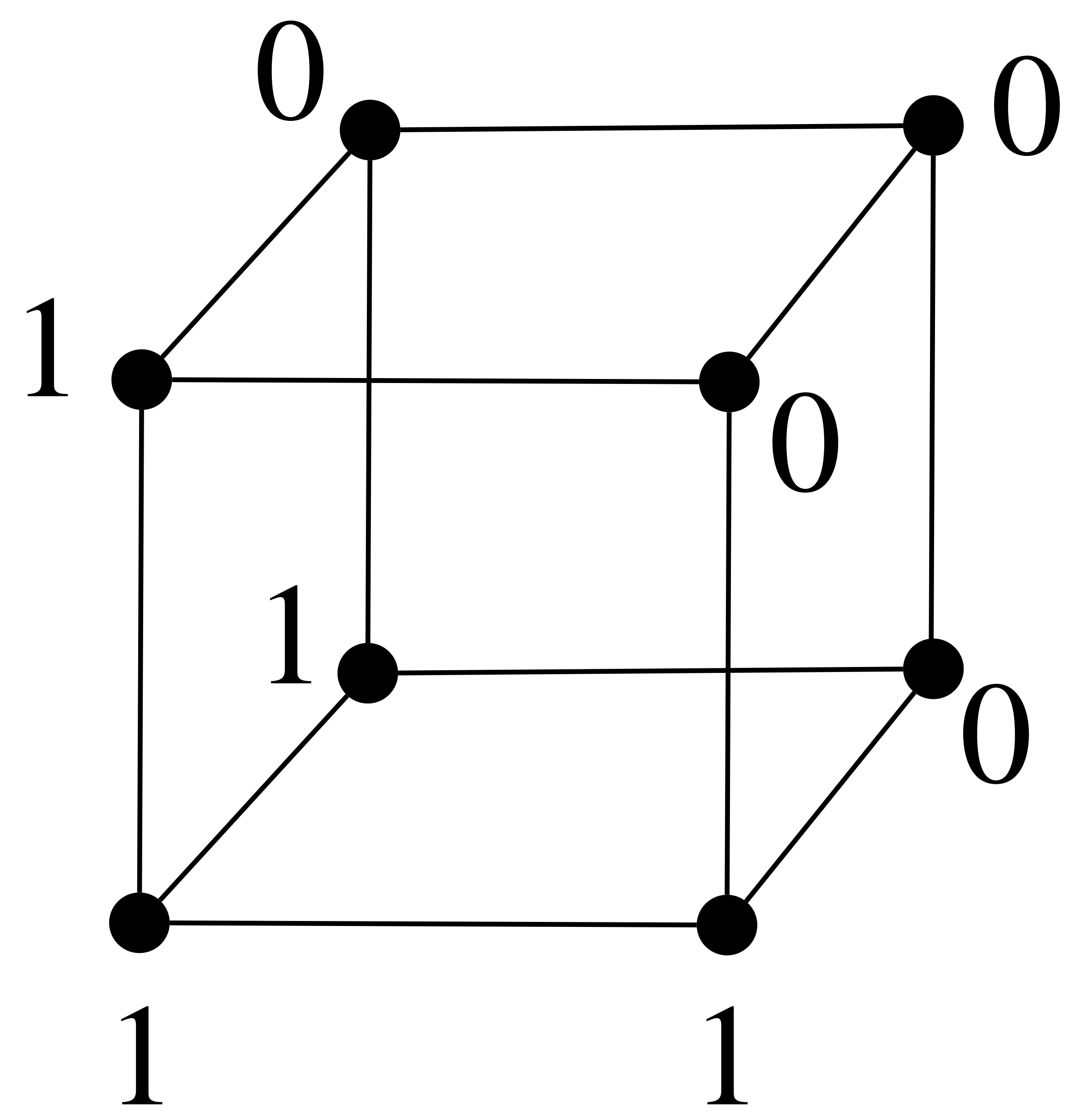

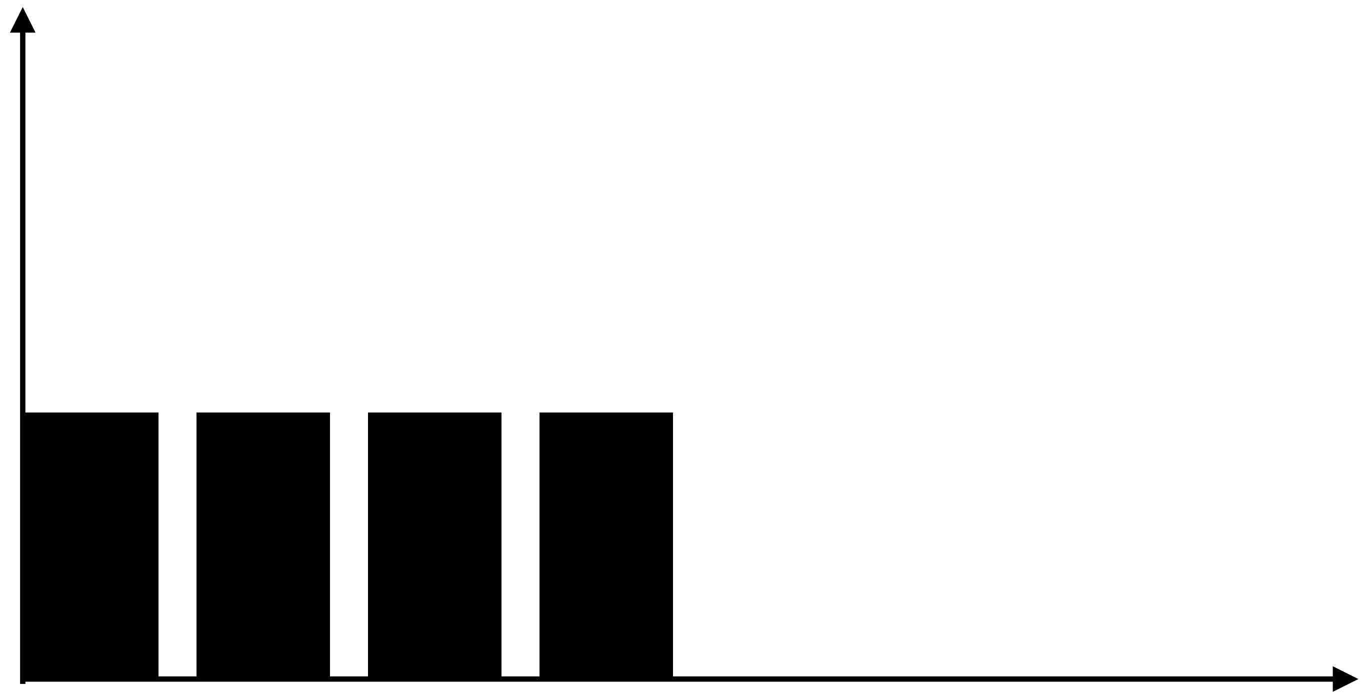

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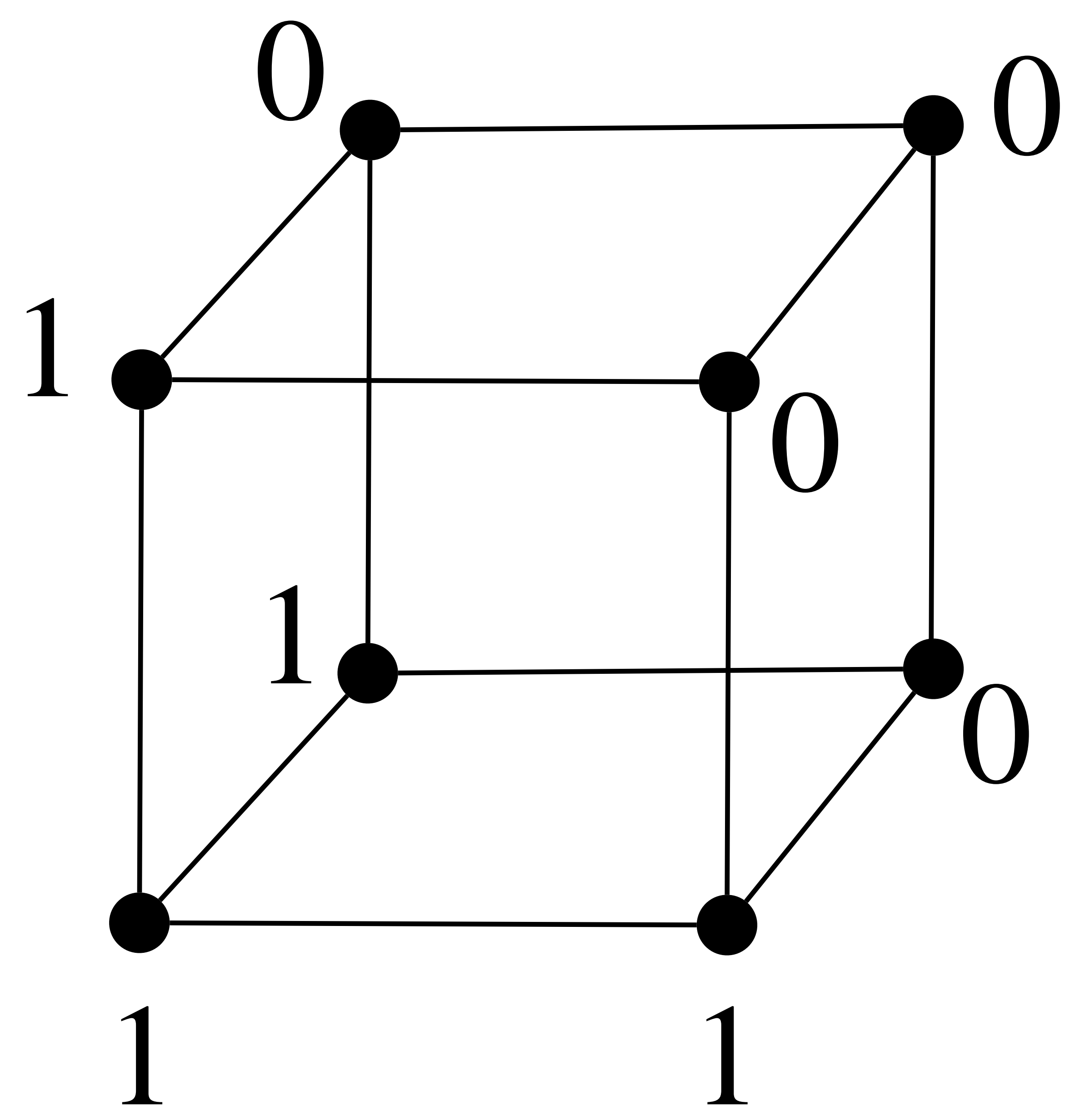

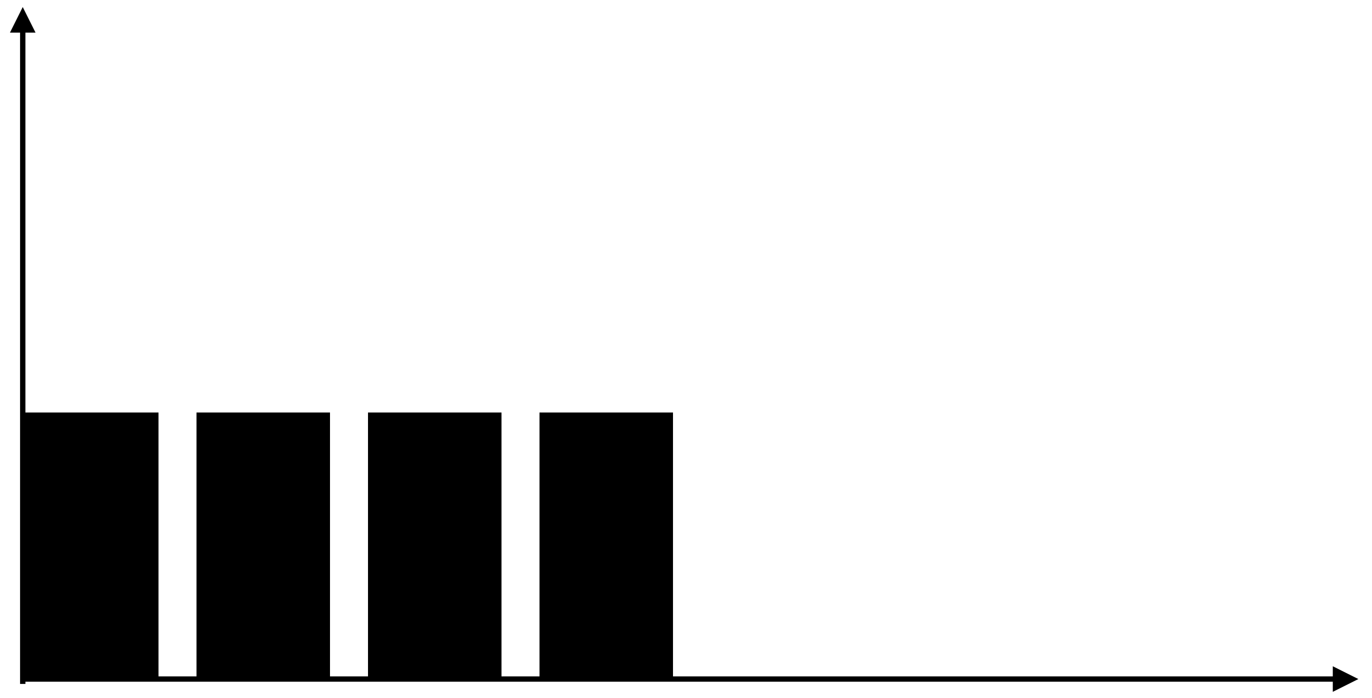

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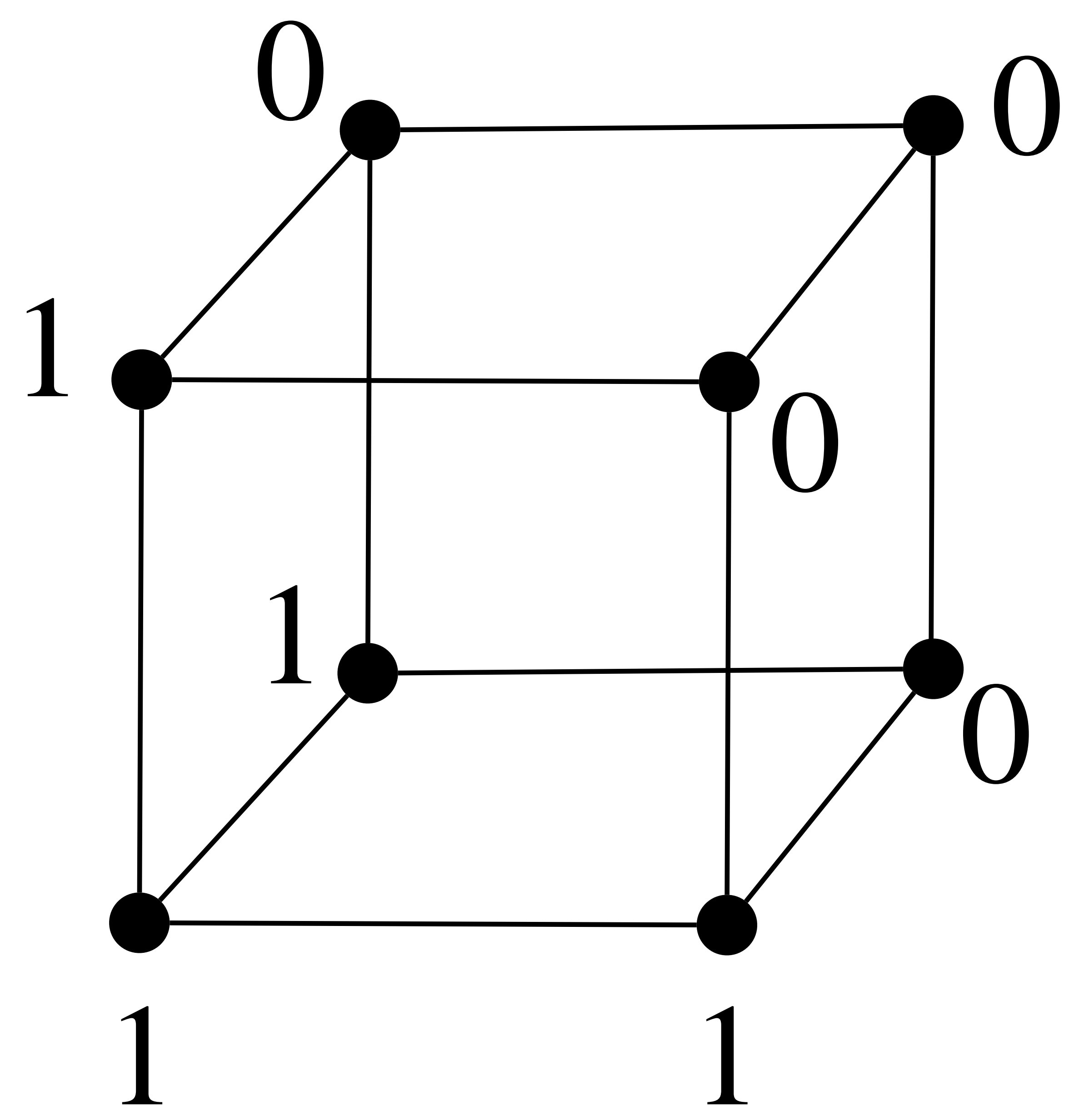

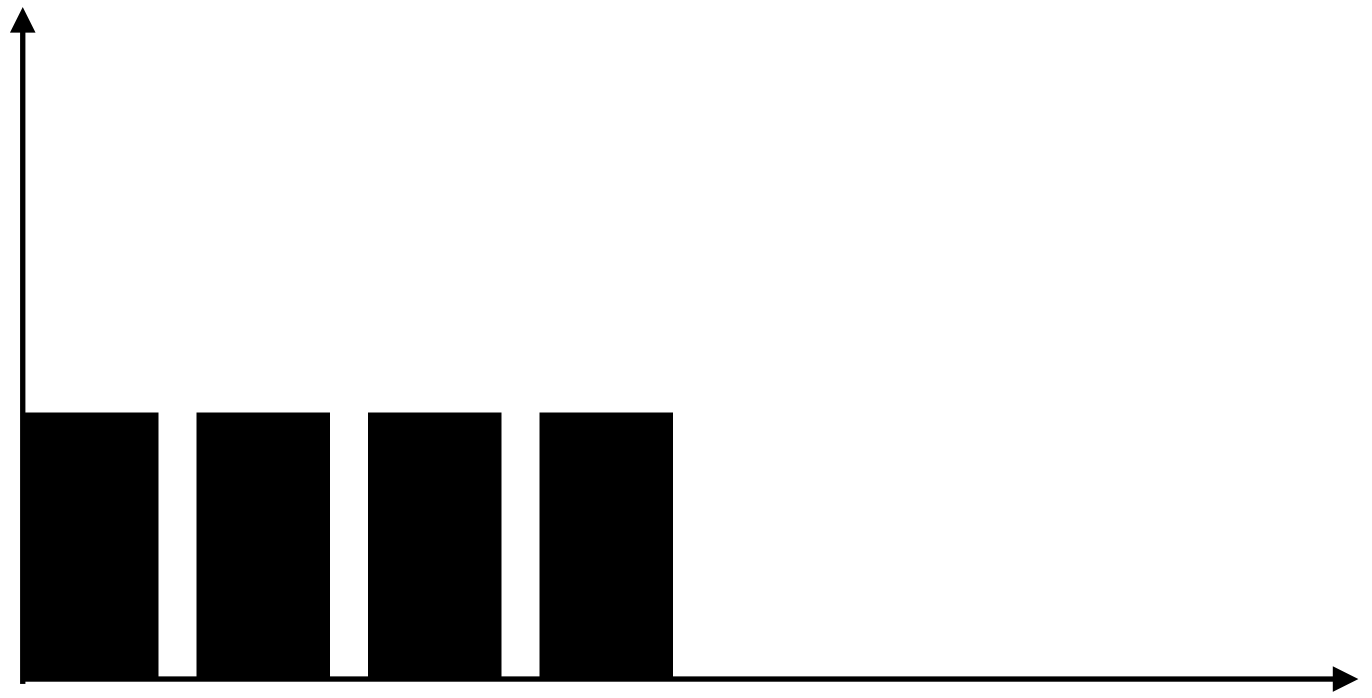

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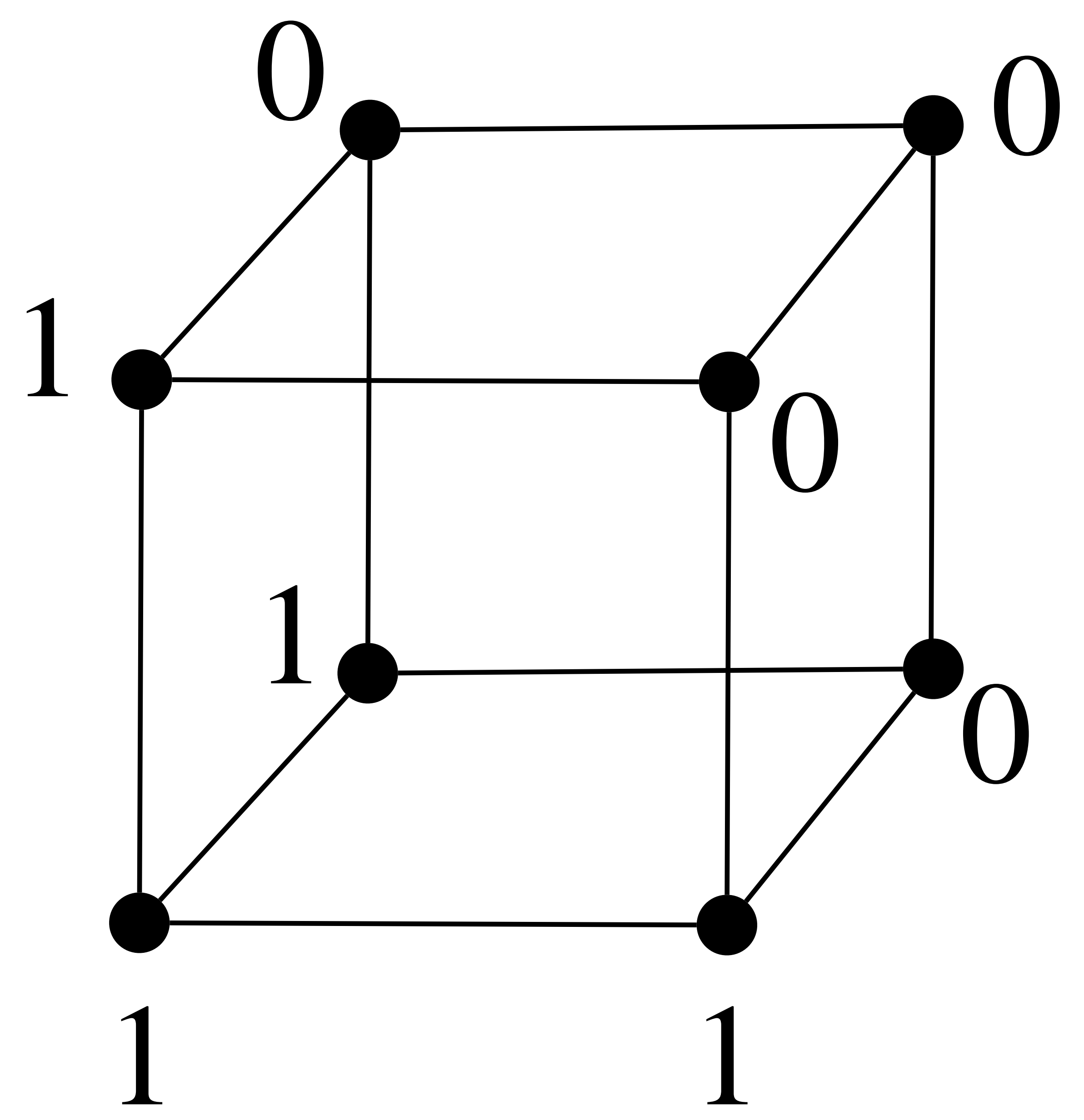

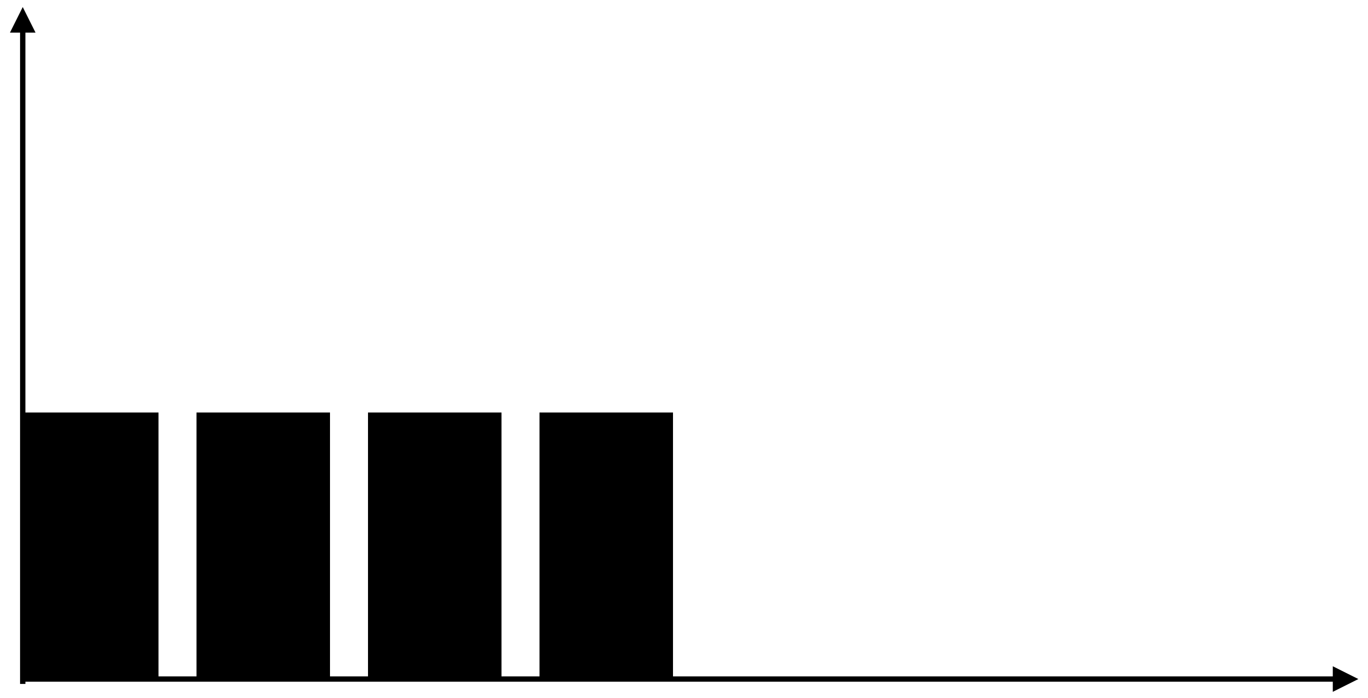

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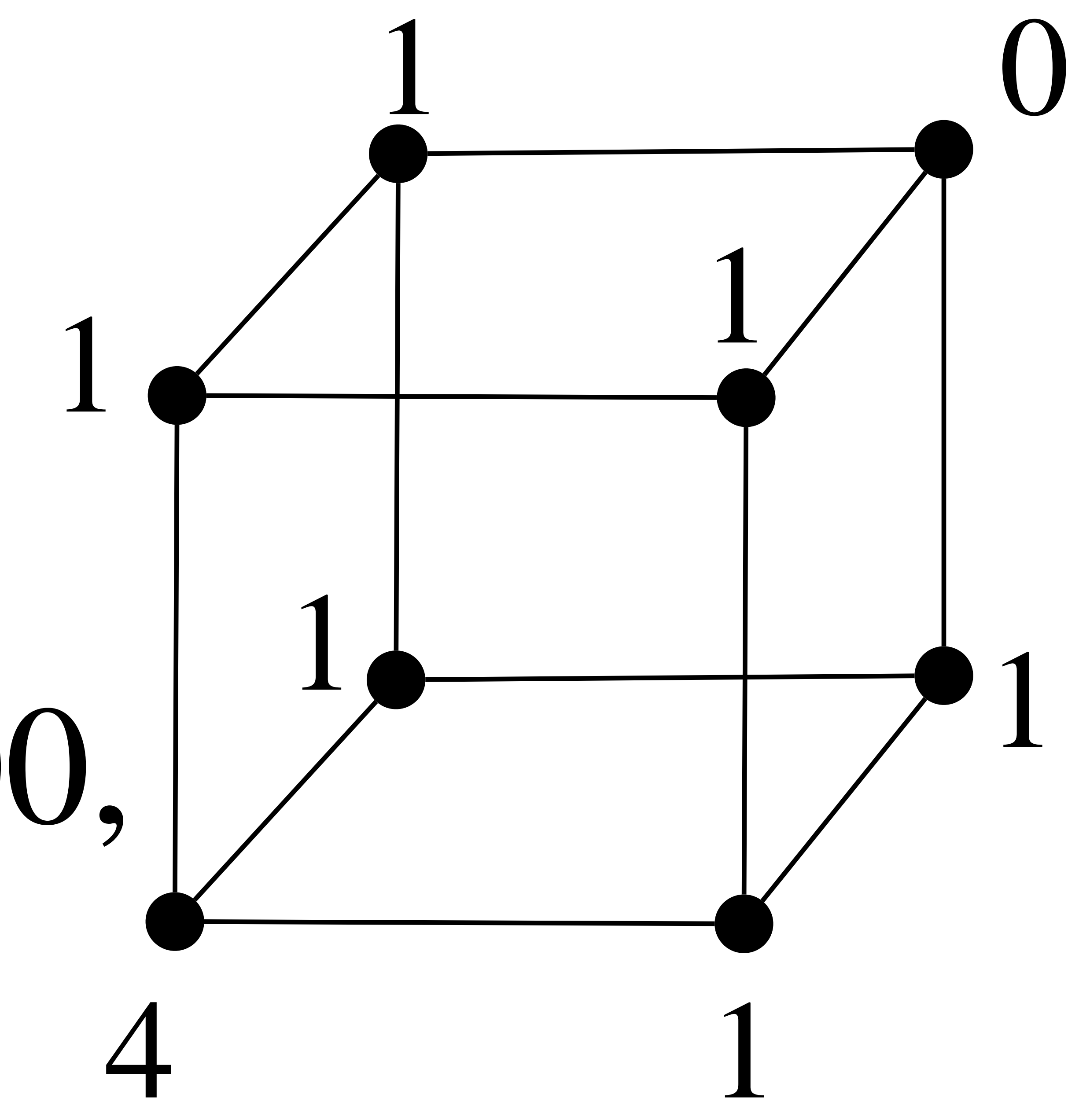

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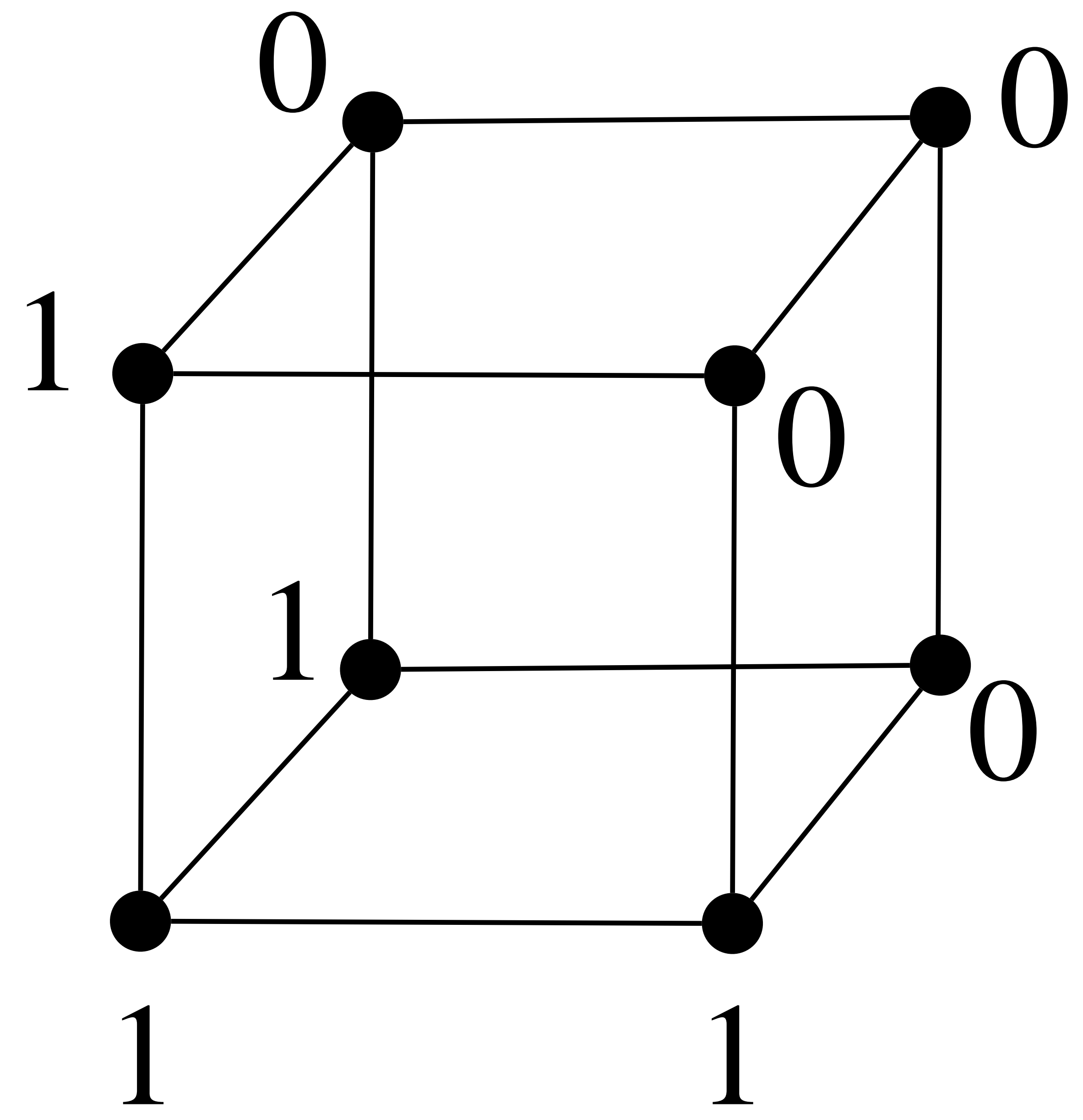

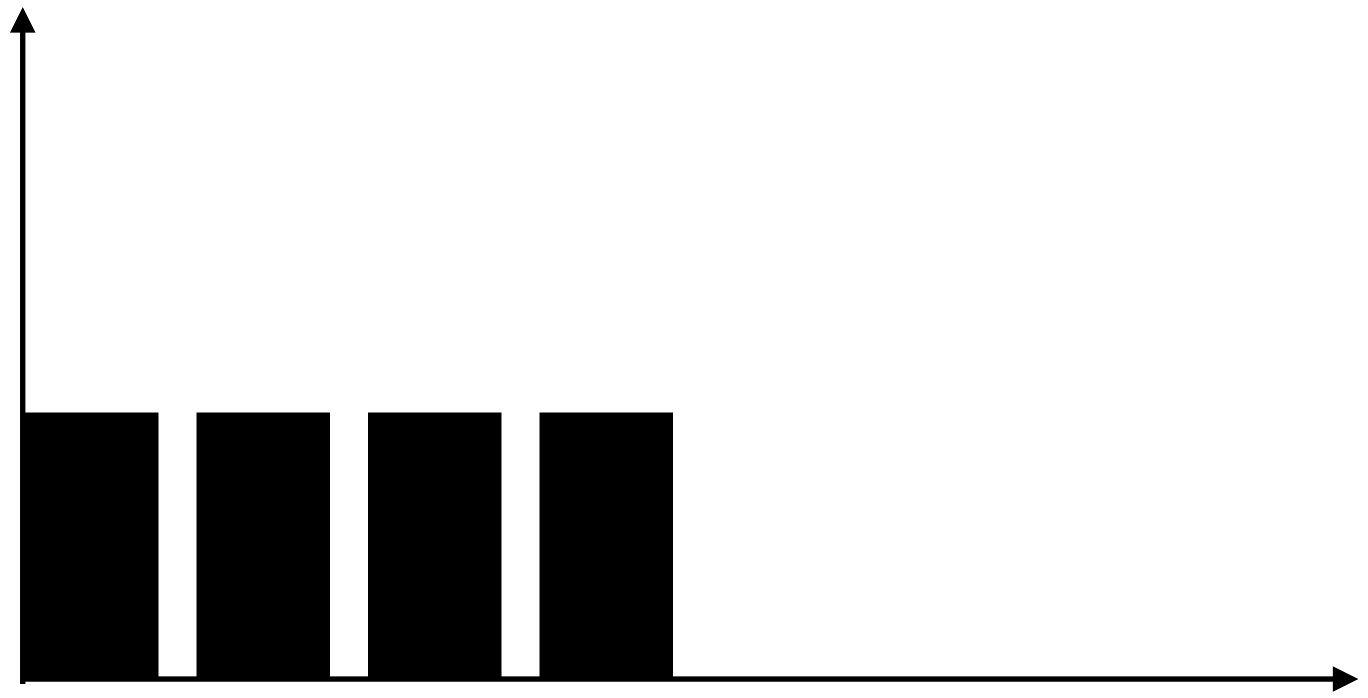

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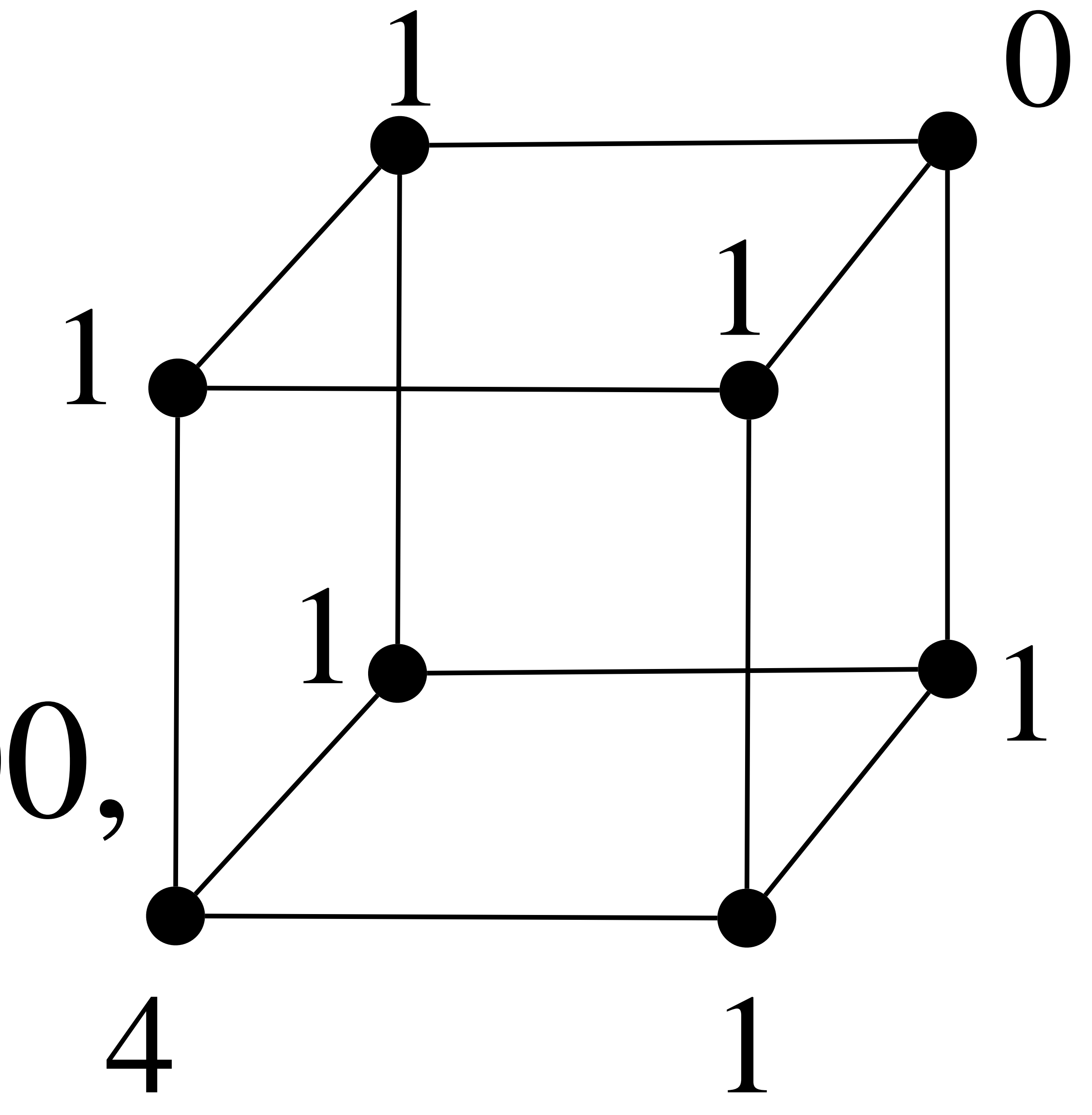

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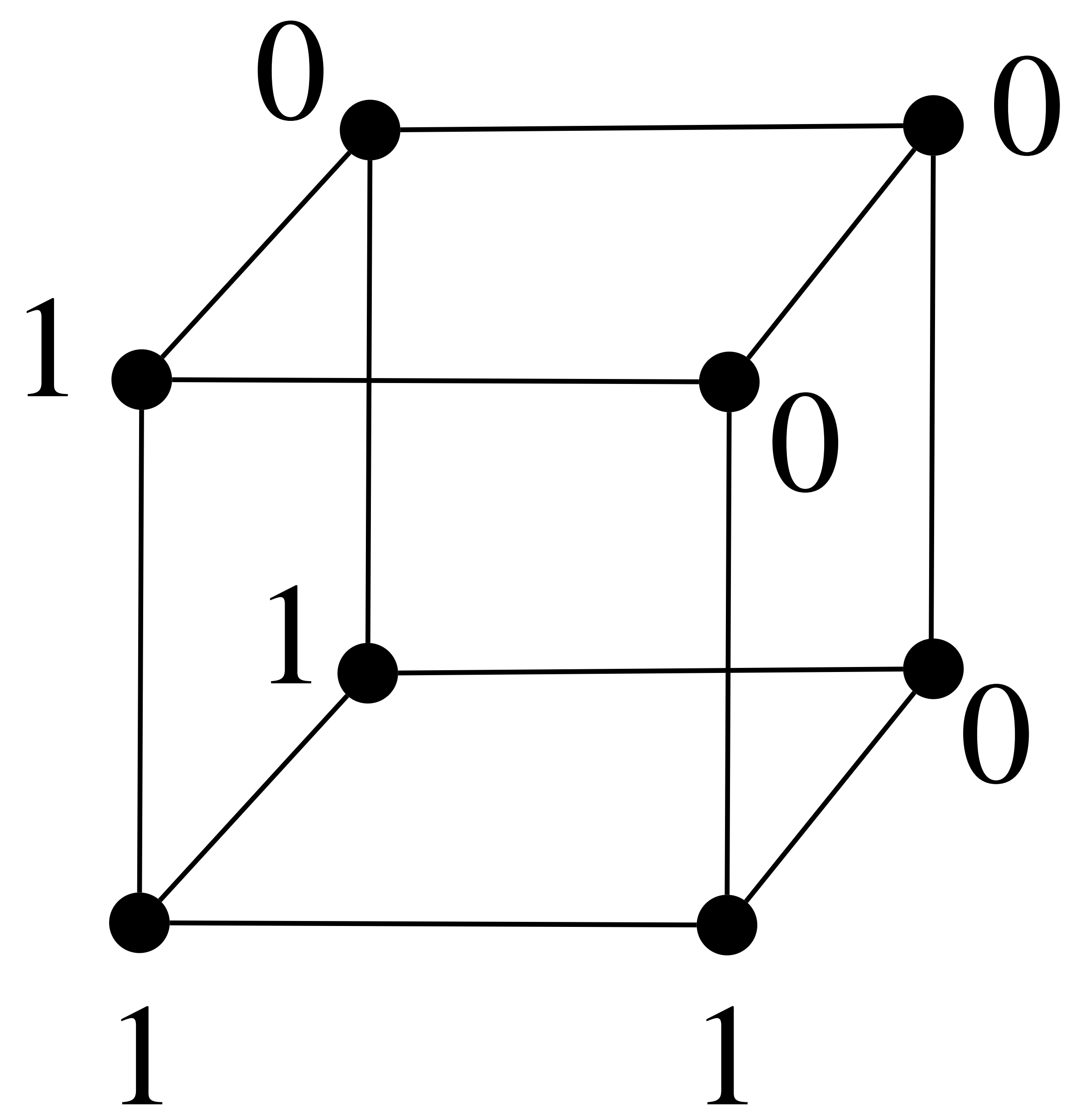

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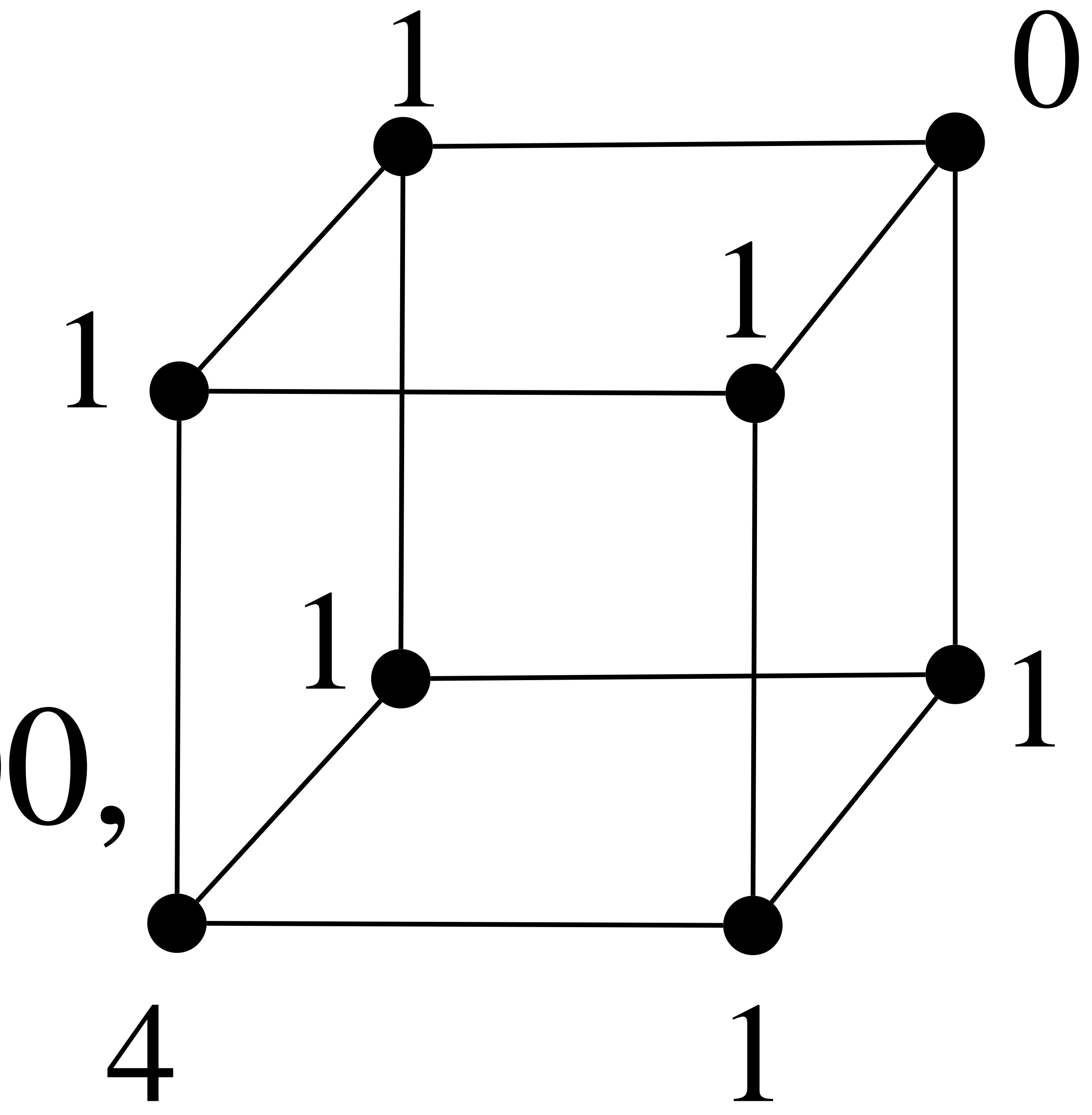

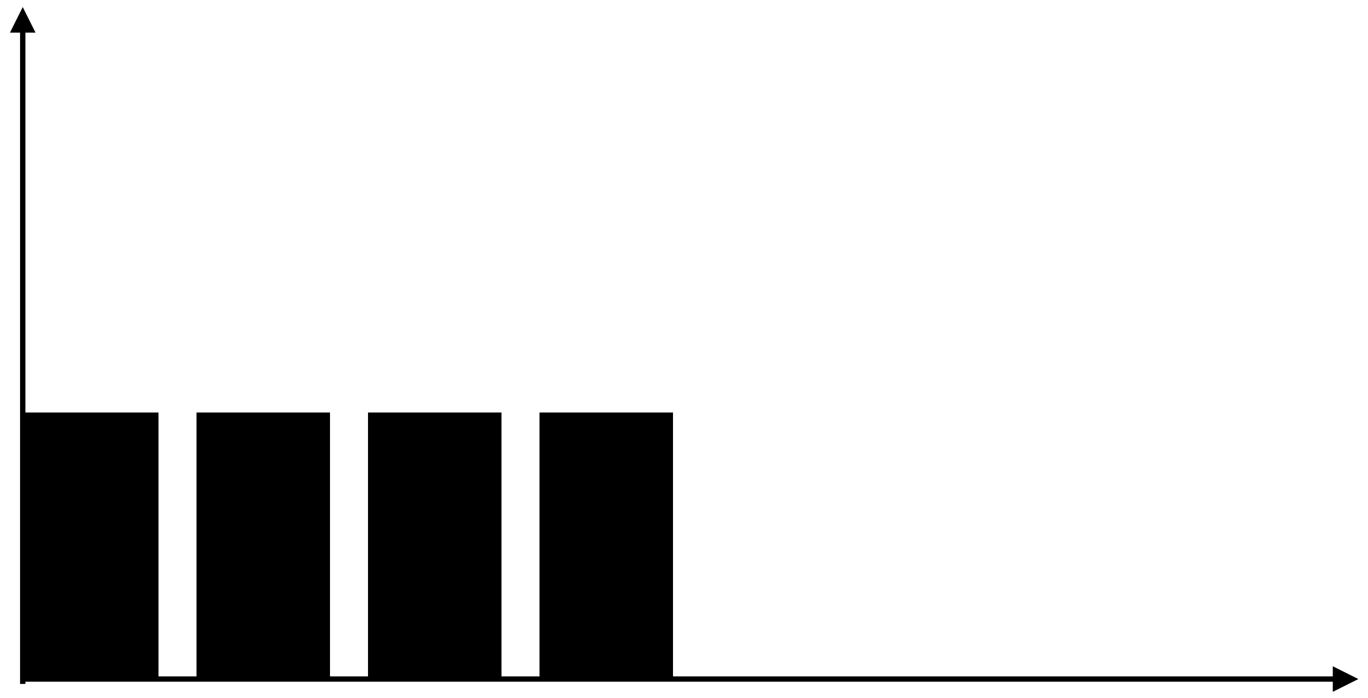

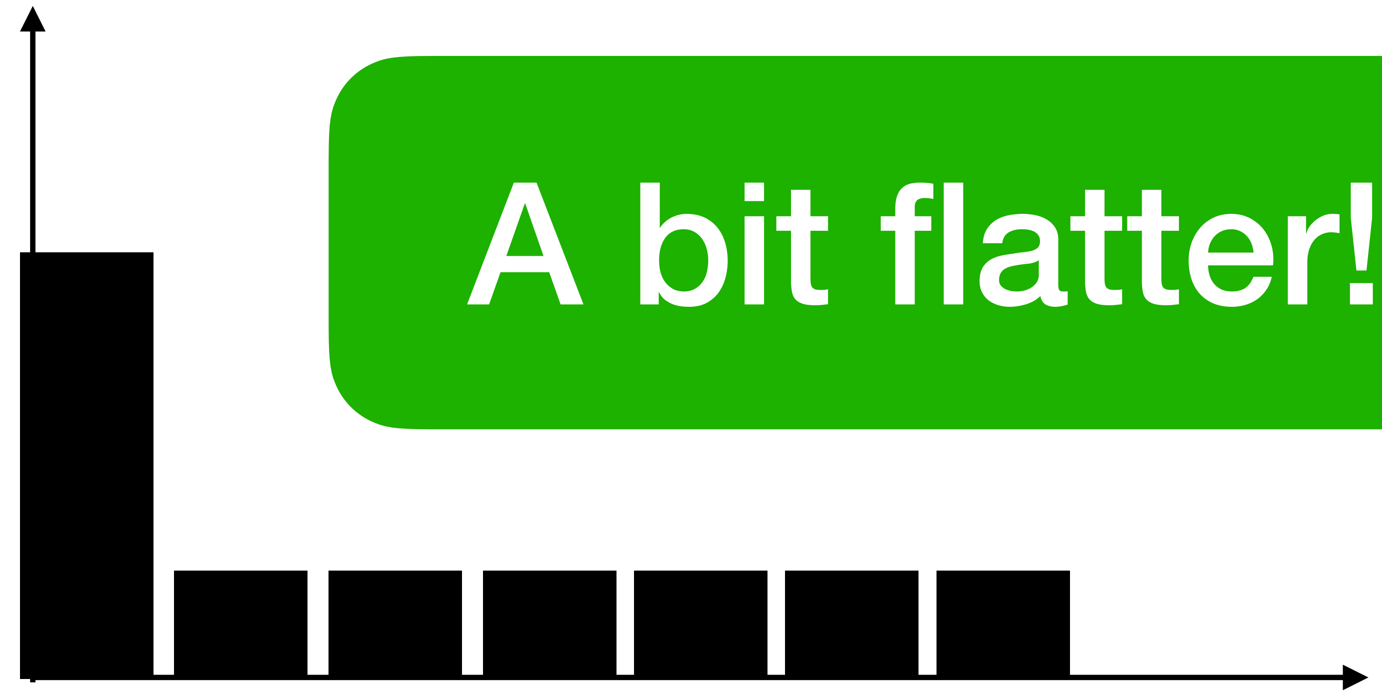

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$\therefore$

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Summary and Open Problems

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Let $\mathscr{B}$ be collection of sets, $\beta = 1 - \frac{1}{n} \log_q \min_{B \in \mathscr{B}} |B|$, C an RLC of rate $\beta + \varepsilon$. Have $|\delta_{C,B}| < q^{-\Omega(\varepsilon n)}$ whp in two cases:

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List-recovery (over $\mathbb{F}_p$), & leakage-resilience!

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- We require constant $\varepsilon > 0$ gap

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Let $\mathbf{C}$ be RLC of rate $1 - h_q(\rho) + \gamma$, and consider RV

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General belief: should have $X_\rho \approx \mathrm{Poisson}(q^{\gamma n})$

assumed in code-based cryptanalysis, e.g., [MT'23]

Poissonness of Weight Distribution?

Let $\mathbf{C}$ be RLC of rate $1 - h_q(\rho) + \gamma$, and consider RV

$$X_\rho := |\mathbf{C} \cap S_\rho|,$$

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Above could hold for both $\gamma > 0$ and $\gamma < 0$

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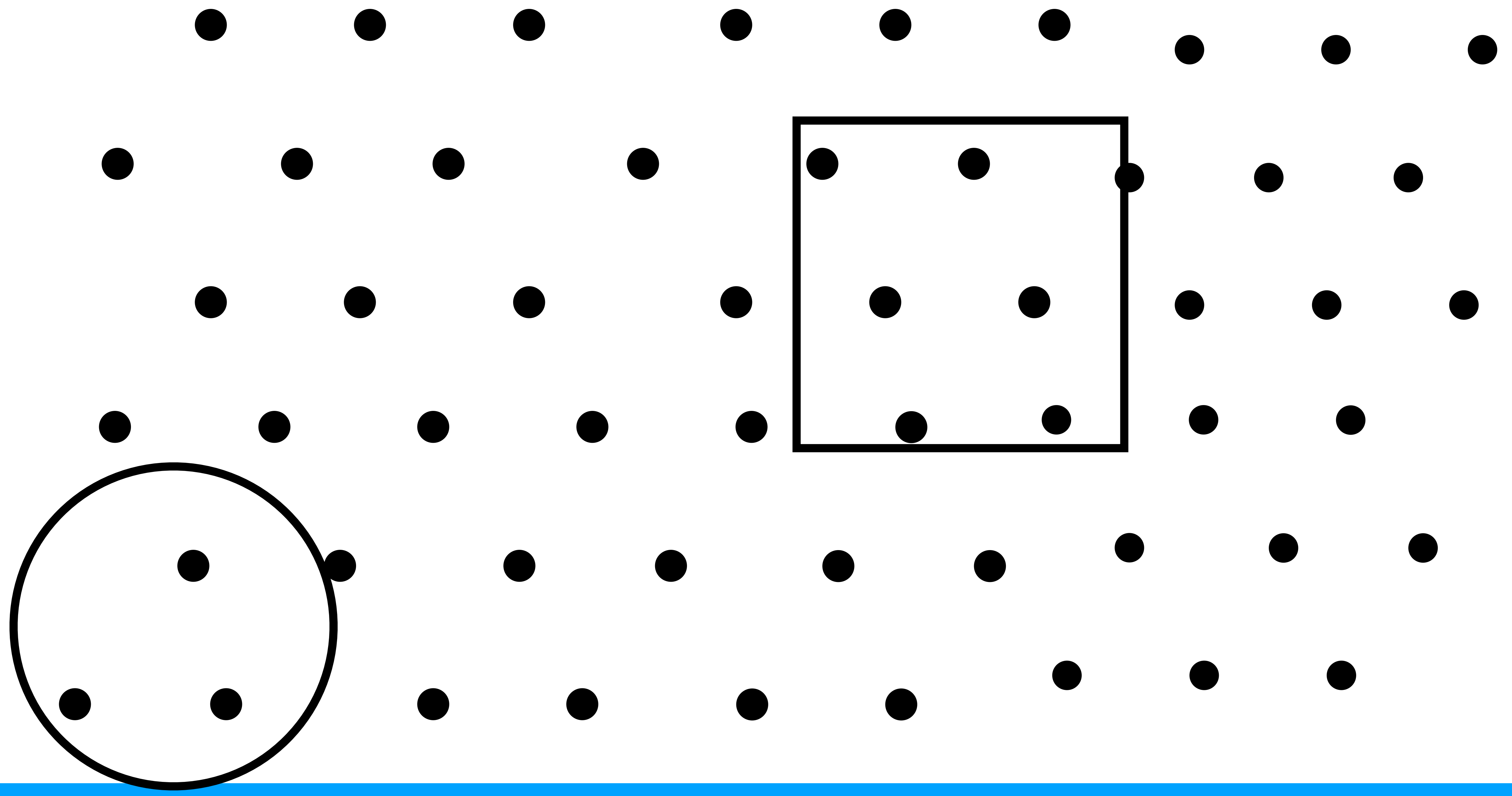
More generally: disjointed landscape for list-decodability of RLC's; is there one unified proof?



Above Capacity

$$R(C) = 1 - \frac{1}{n}\log_q|B| + \varepsilon, \text{ so}$$

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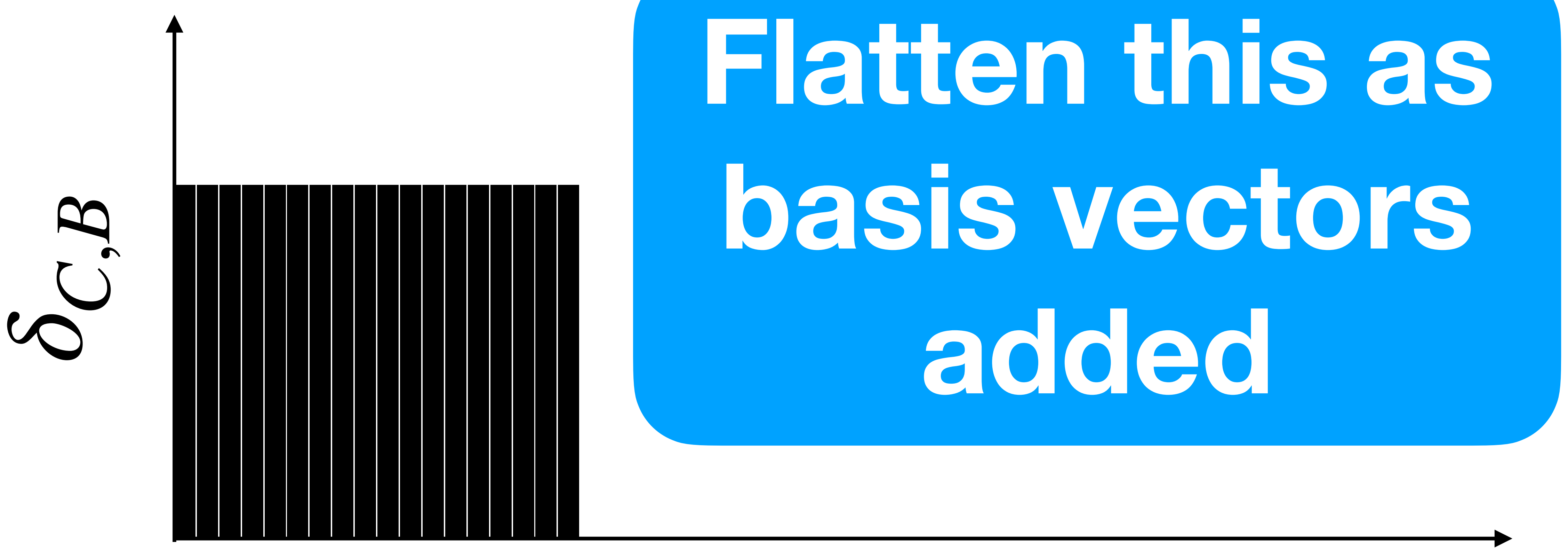
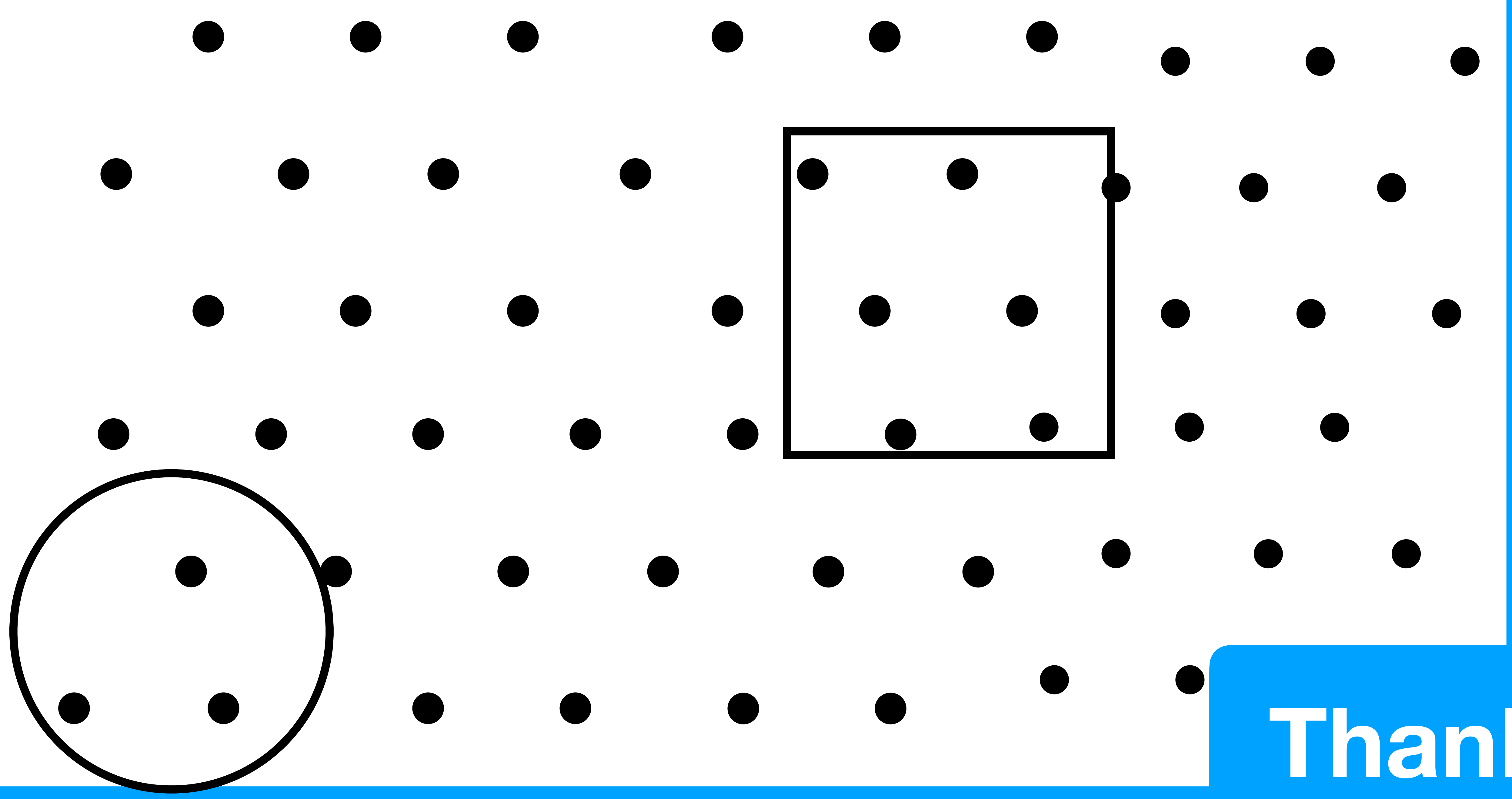
Strategy



Main Proof Idea

Transfer smoothness from $B - B$ to B



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Thank you!
Questions?

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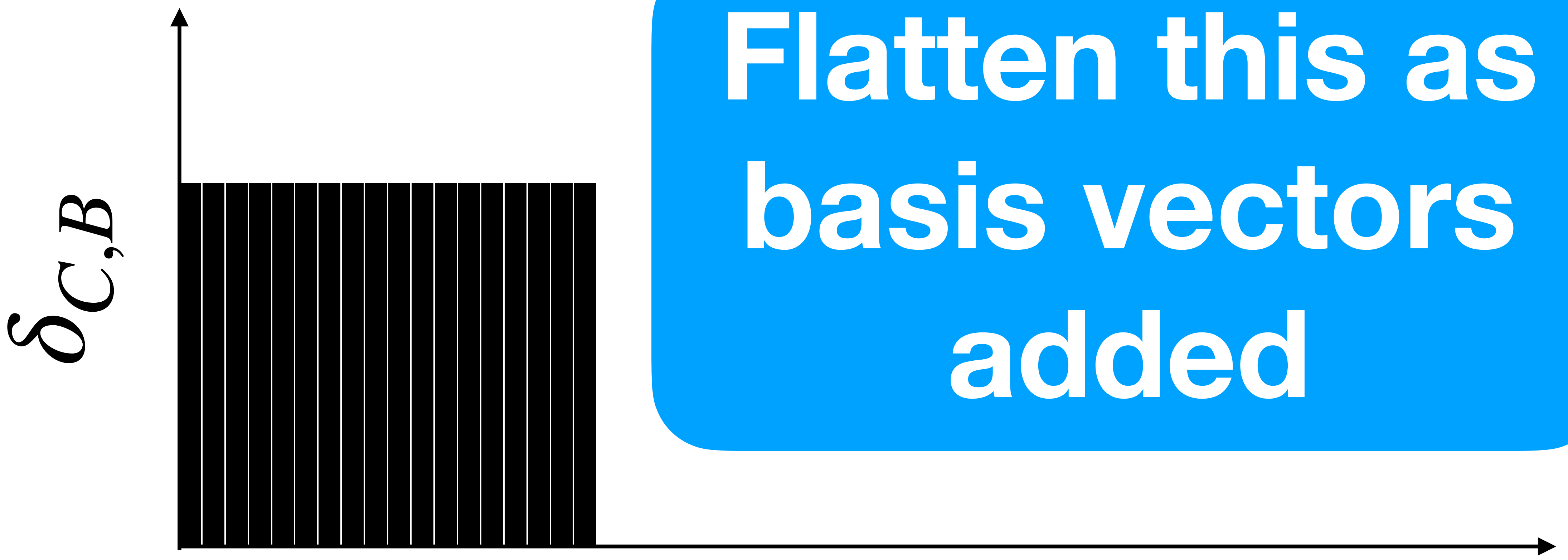

Flatten this as basis vectors added

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