Improved capacity upper bounds for the deletion channel using a parallelized Blahut-Arimoto Algorithm

Martim Pinto

Técnico-ULisboa

João Ribeiro

Inst. Telecomunicações
+ Técnico-ULisboa

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A simple model of errors that cause loss of synchronization.

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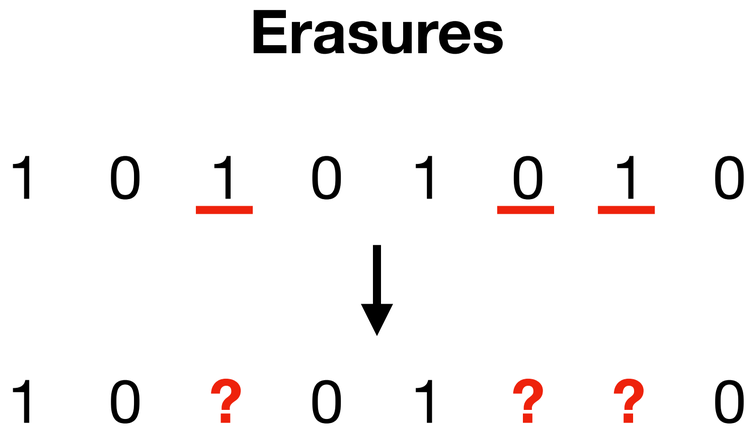

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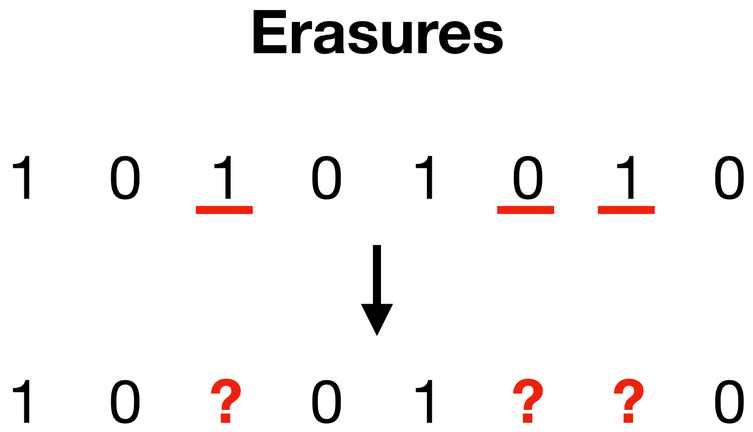

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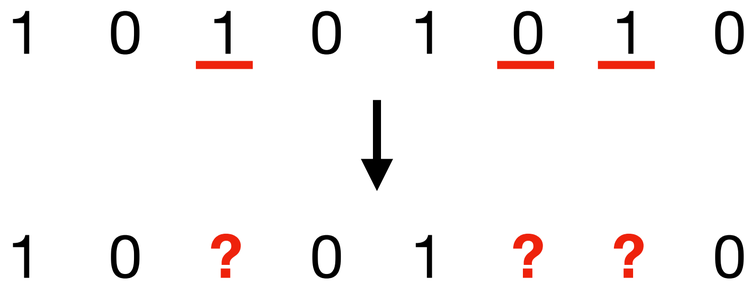

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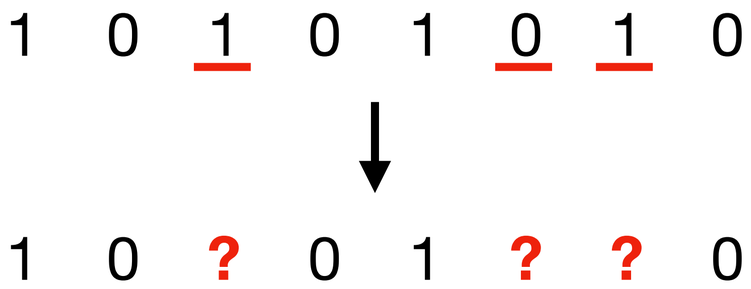

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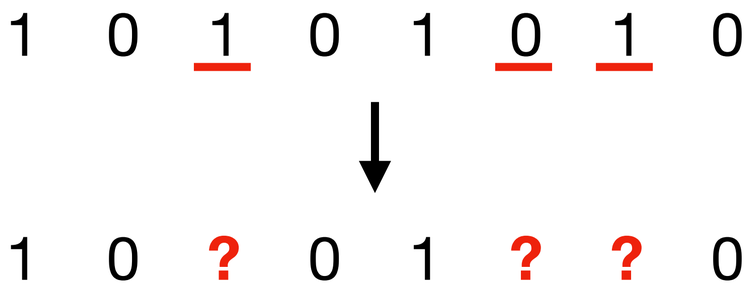

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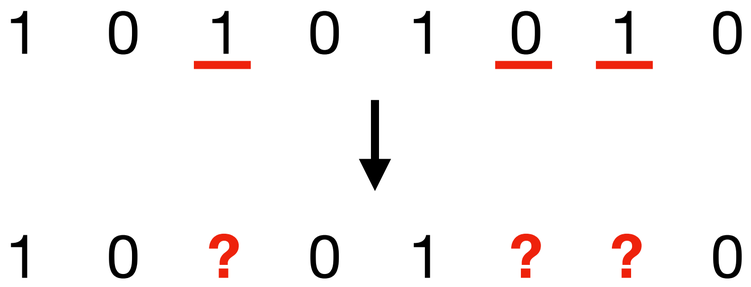

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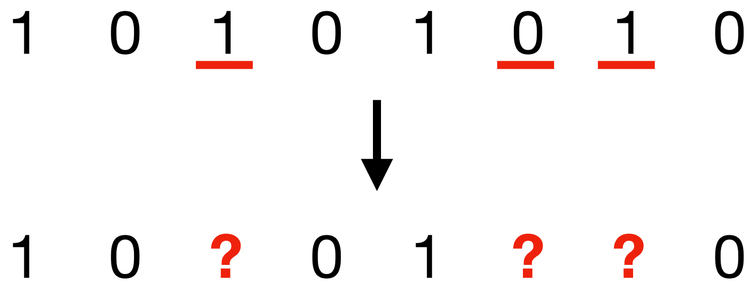

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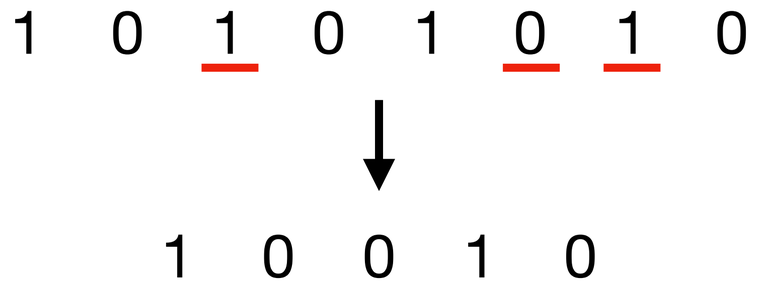

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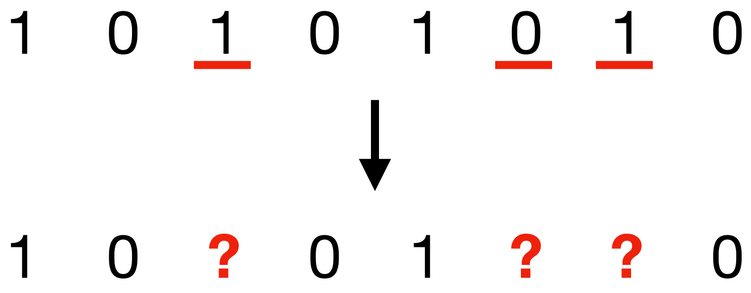

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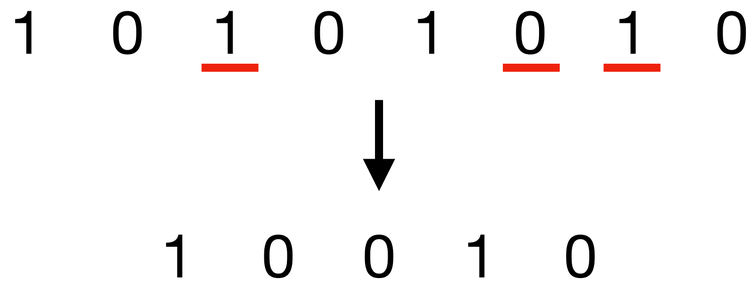

New approaches required!

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but doesn't simplify like for DMCs…

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- Non-trivial **upper bounds.**
  [Diggavi-Mitzenmacher-Pfister, Fertonani-Duman, Kalai-Mitzenmacher-Sudan, Kanoria-Montanari, Rahmati-Duman, Cheraghchi, Rubinstein-Con]

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0 0 1 **1** 1 **0** 1 0 0 $\longrightarrow$ **x** 1 1 **x** 1 0 0

$\Longrightarrow$ channel output becomes memoryless across input runs.

$\Longrightarrow$ can upper bound $C(d)$ by upper bounding capacity per unit cost of the DMC

$$\ell \in \mathbb{N} \mapsto \mathsf{Bin}(\ell, 1-d)$$

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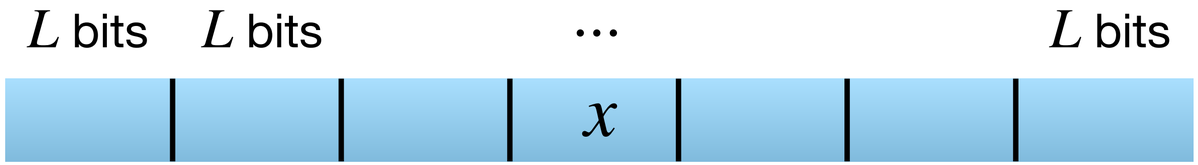

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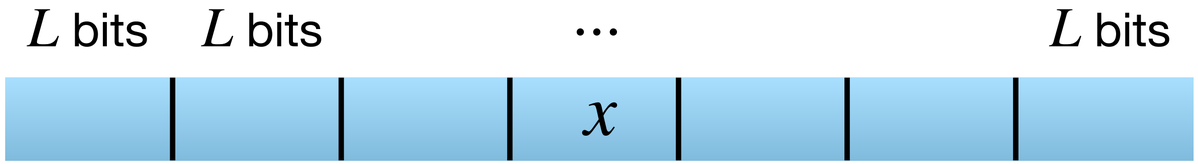

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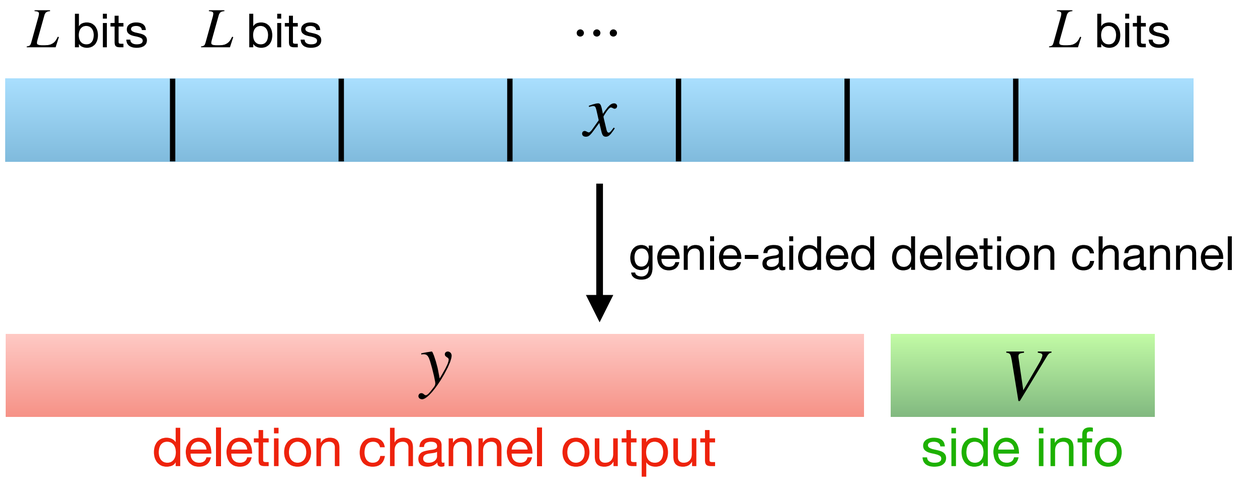

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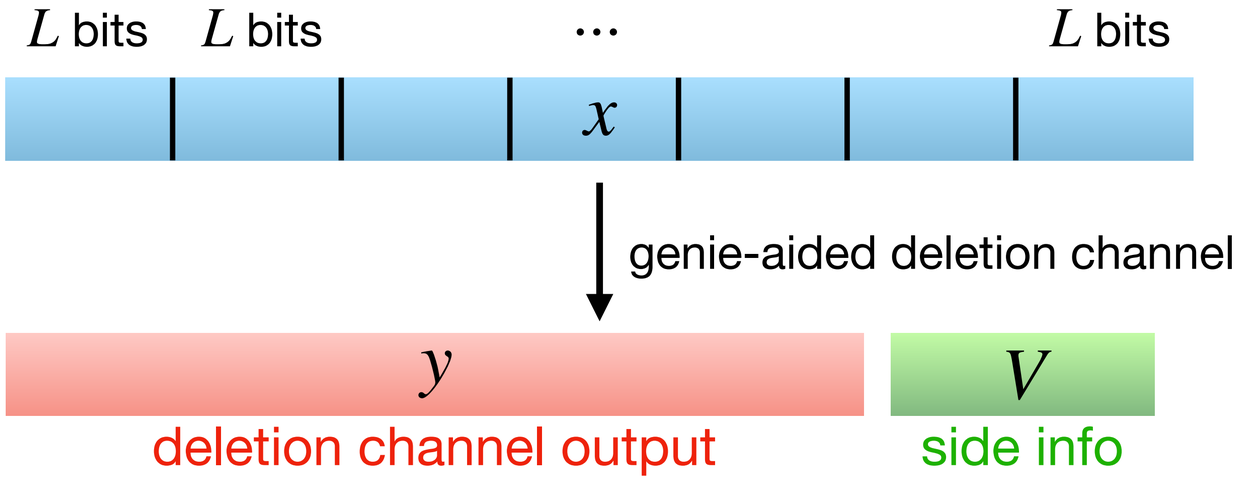

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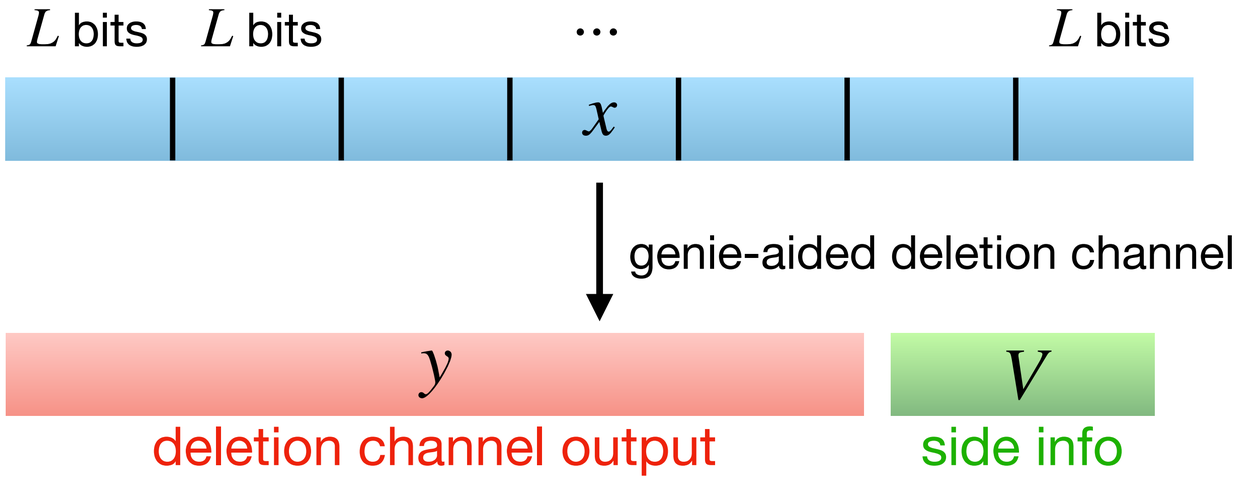

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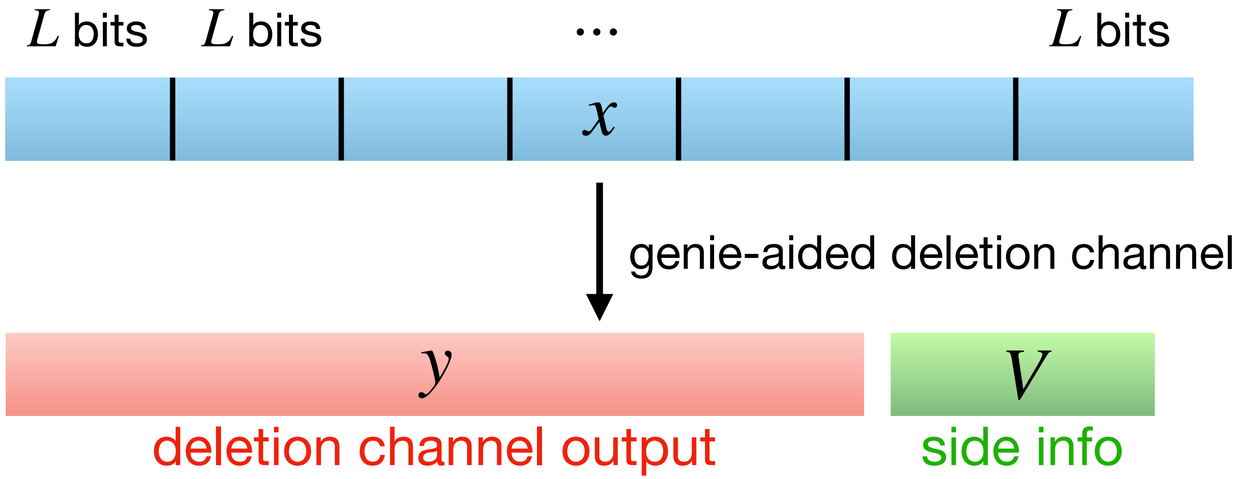

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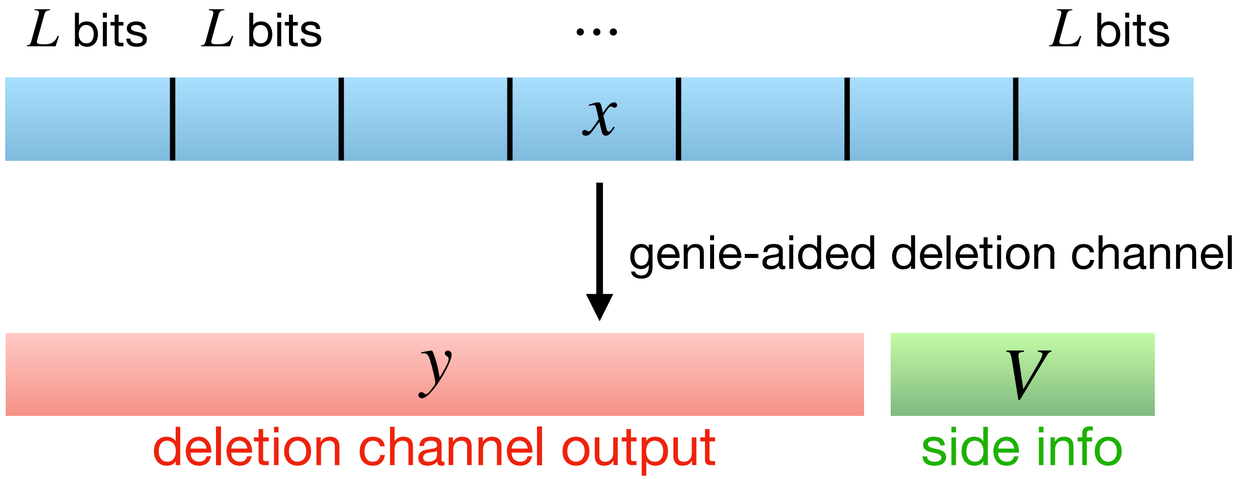

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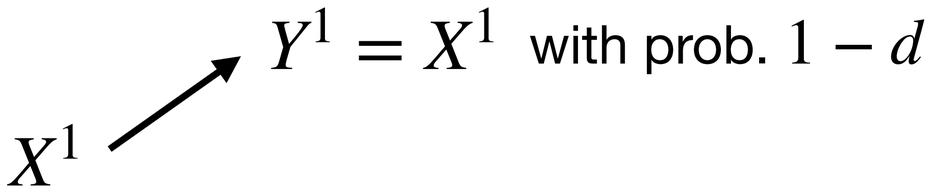

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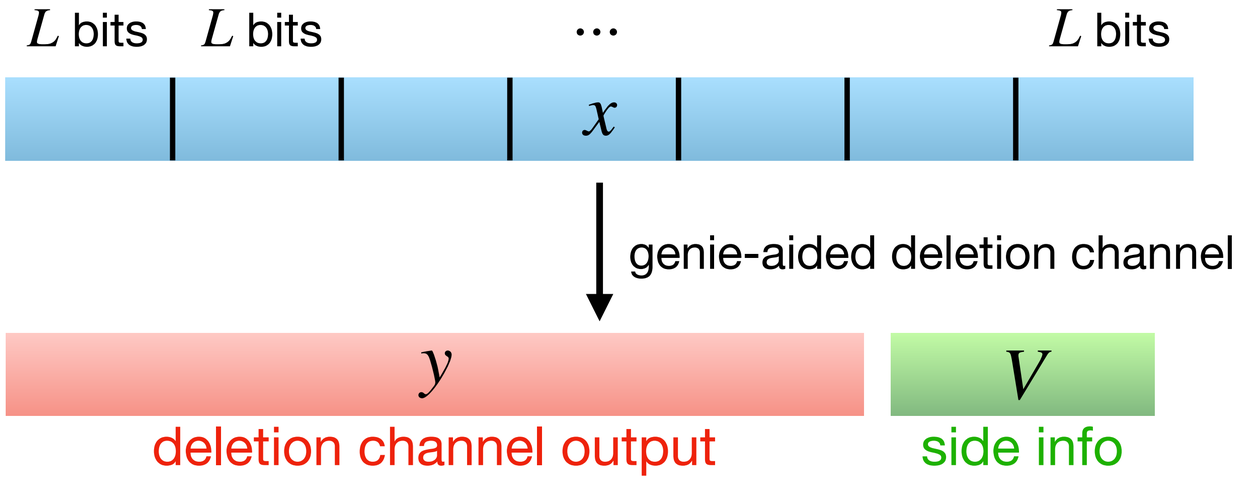

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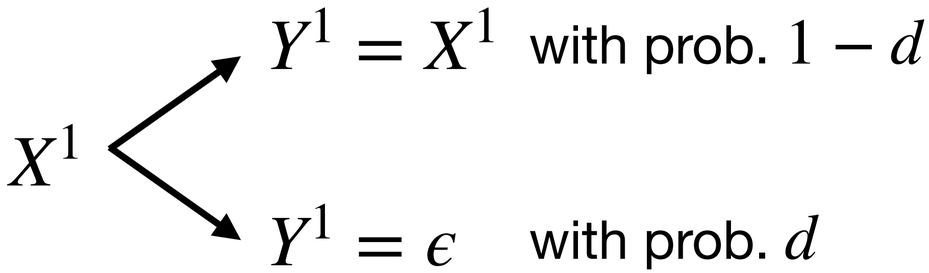

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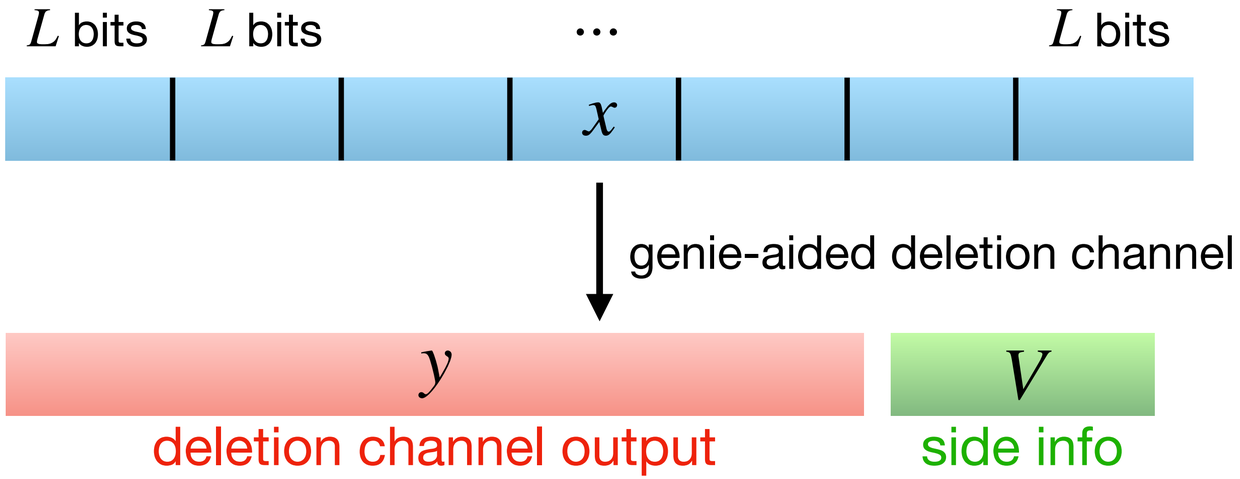

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$$X^1 \begin{cases} Y^1 = X^1 & \text{with prob. } 1-d \\ Y^1 = \epsilon & \text{with prob. } d \end{cases}$$

$$\implies C(d) \leq \sup_{X^1} I(X^1; Y^1) = 1 - d$$

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Prior work:

Fertonani-Duman 2010	Computed all $C_{L,k}$'s for $L = 17$ and some $C_{L,k}$'s up to $L = 22$.
Rubinstein-Con 2023	Computed all $C_{L,k}$'s for $L = 22$ and some $C_{L,k}$'s up to $L = 28$.

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So the Blahut-Arimoto algorithm will require time and space exponential in L…

Prior work:

Fertonani-Duman 2010	Computed all $C_{L,k}$'s for $L = 17$ and some $C_{L,k}$'s up to $L = 22$.
Rubinstein-Con 2023	Computed all $C_{L,k}$'s for $L = 22$ and some $C_{L,k}$'s up to $L = 28$.

Improved BAA implementation via time-memory tradeoffs for computing channel transition probabilities

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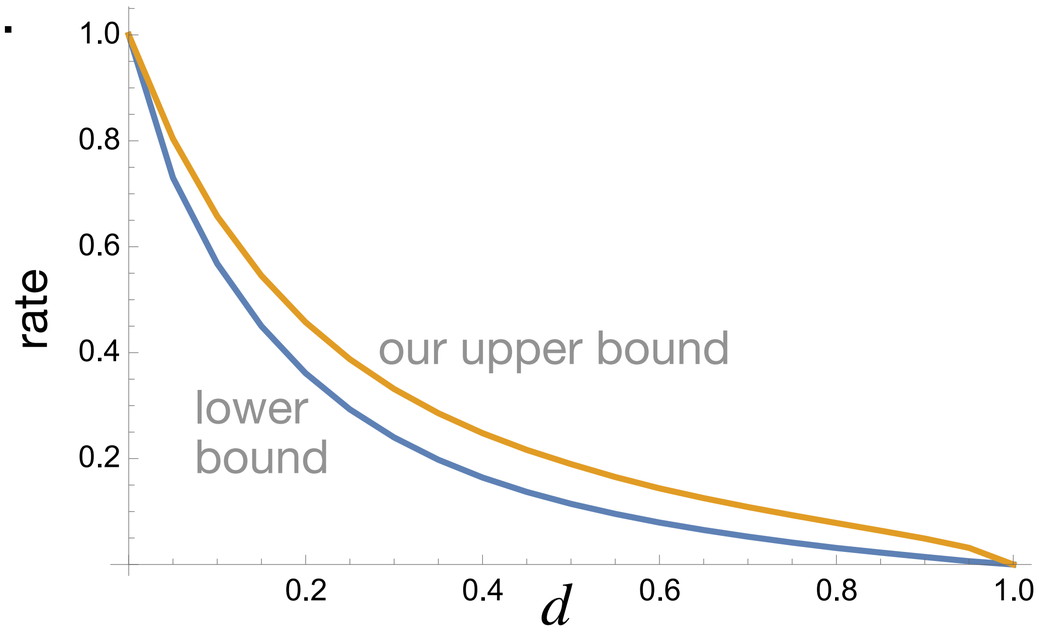

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3. Repeat with W in place of P_X until “error” is small enough.

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Each step in an iteration can be parallelized in a very simple way!

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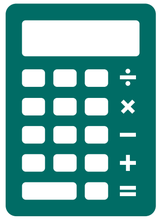
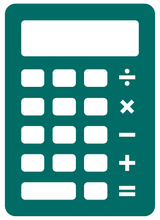
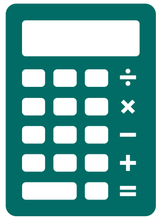

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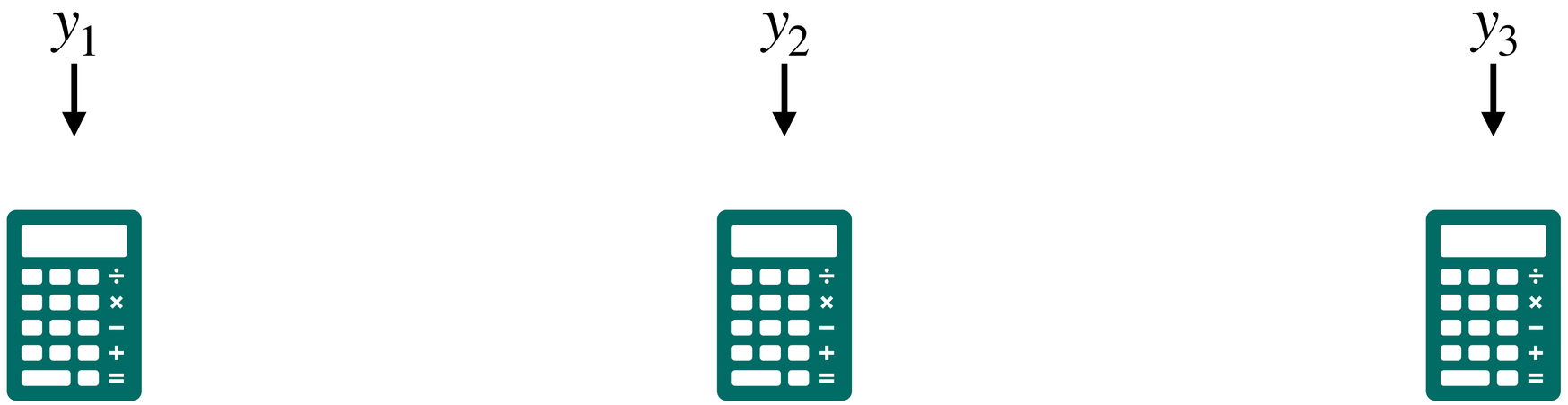

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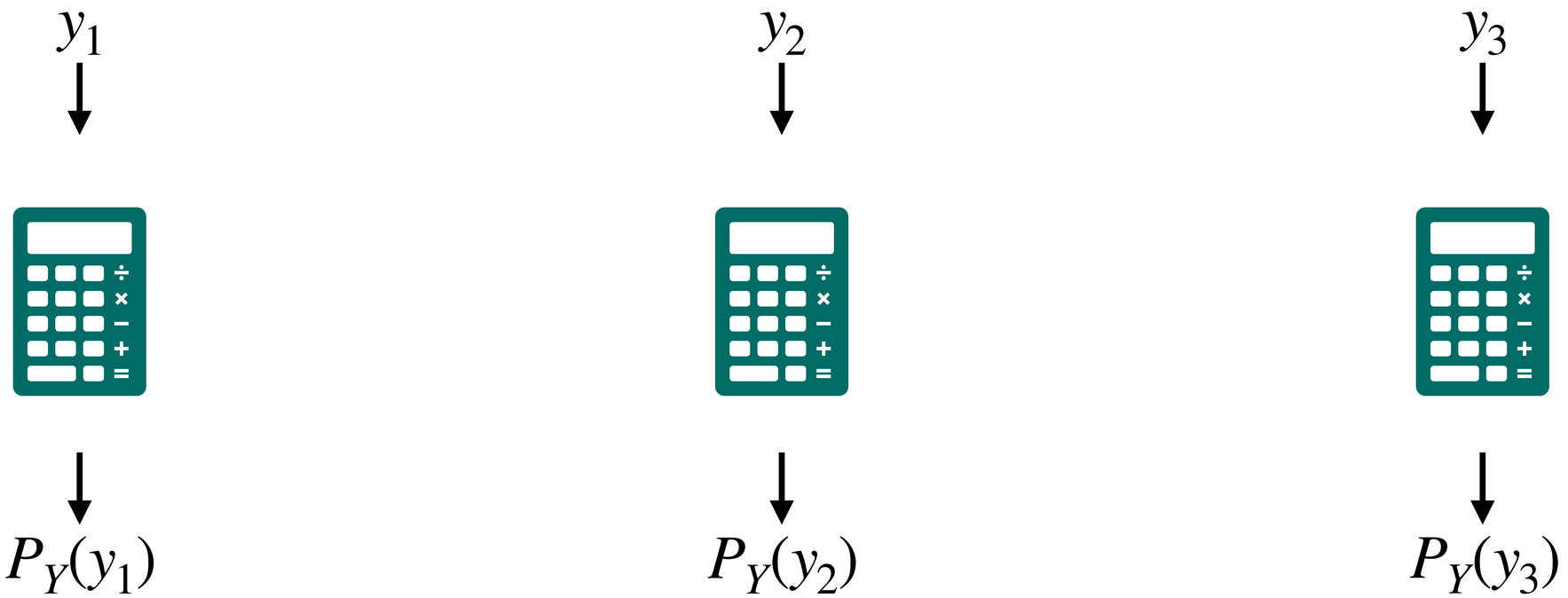

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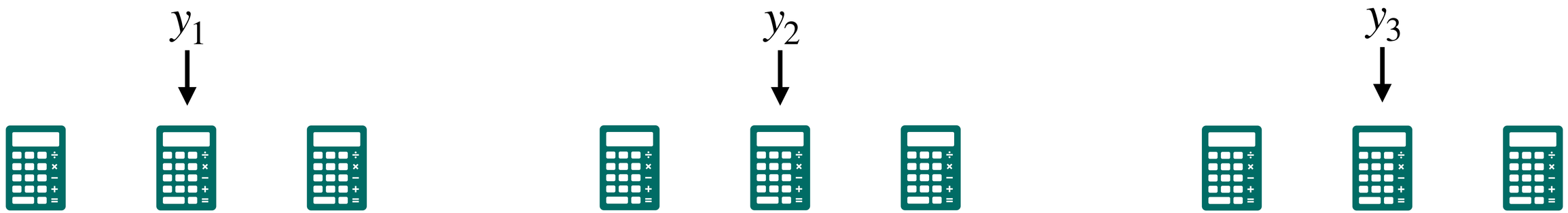

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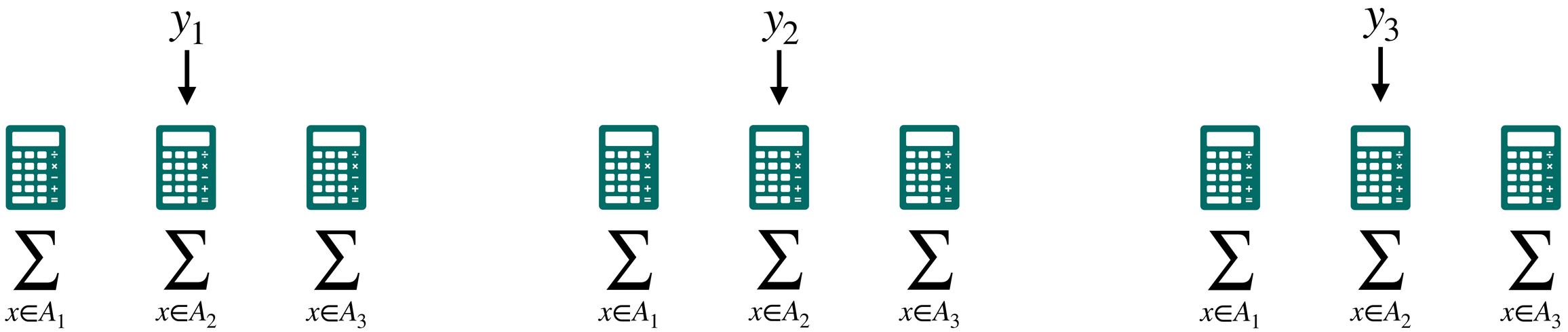

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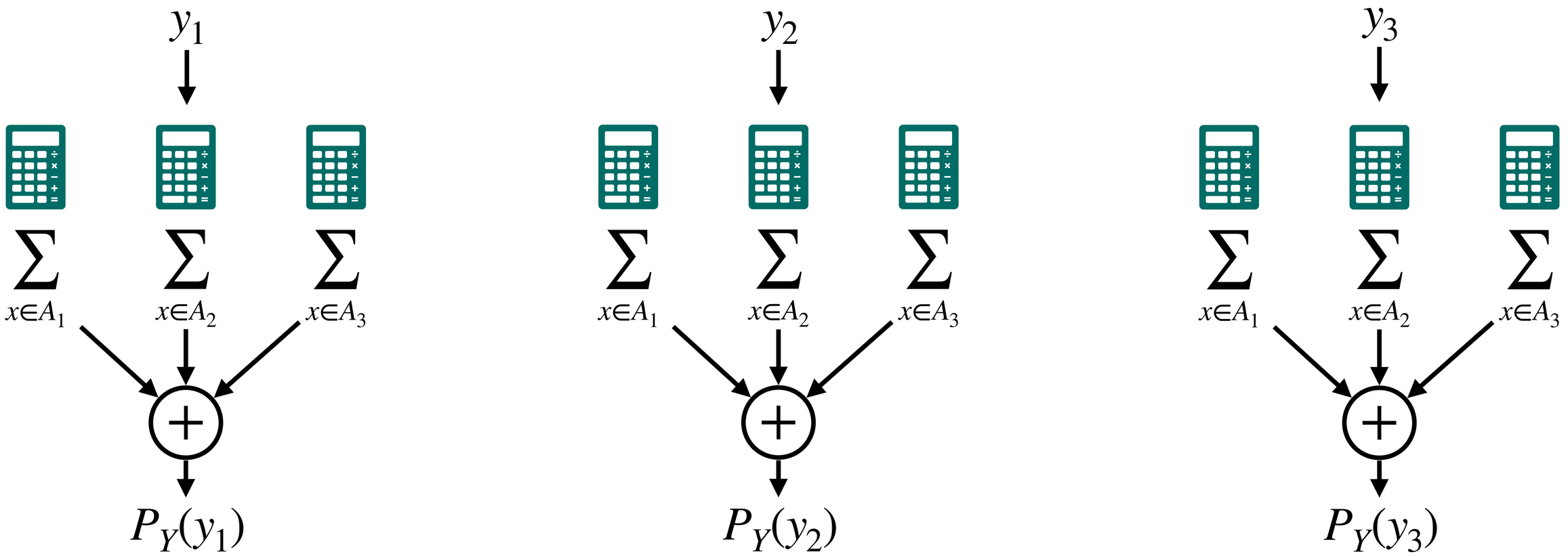

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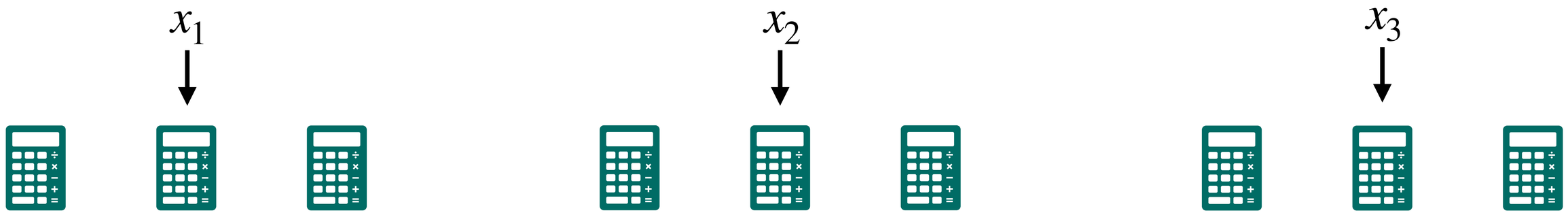

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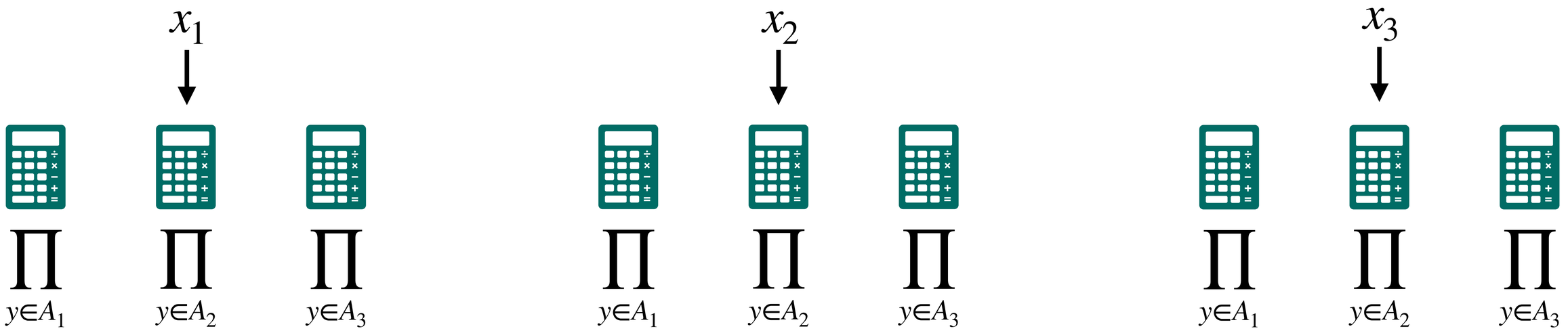

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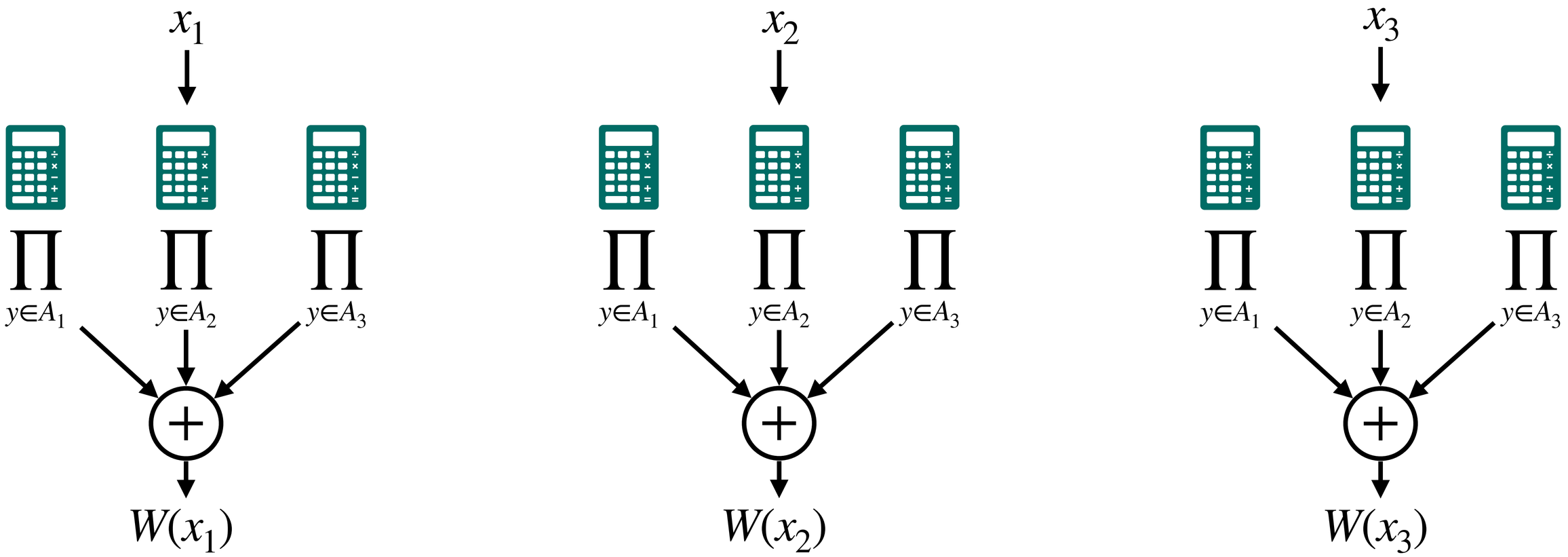

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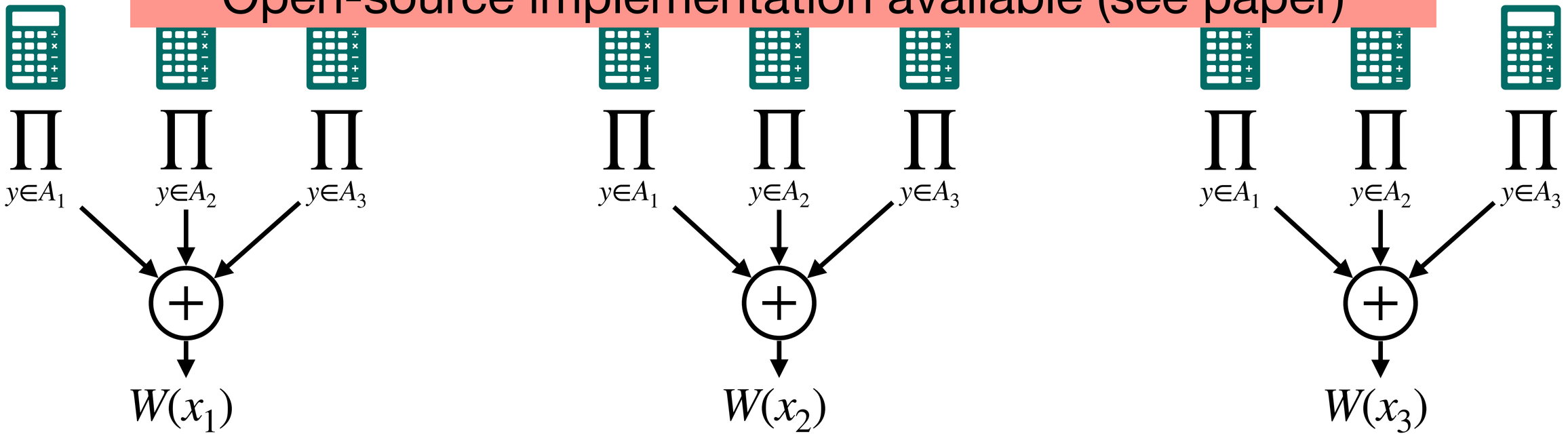
These steps fit nicely into GPUs!

Open-source implementation available (see paper)



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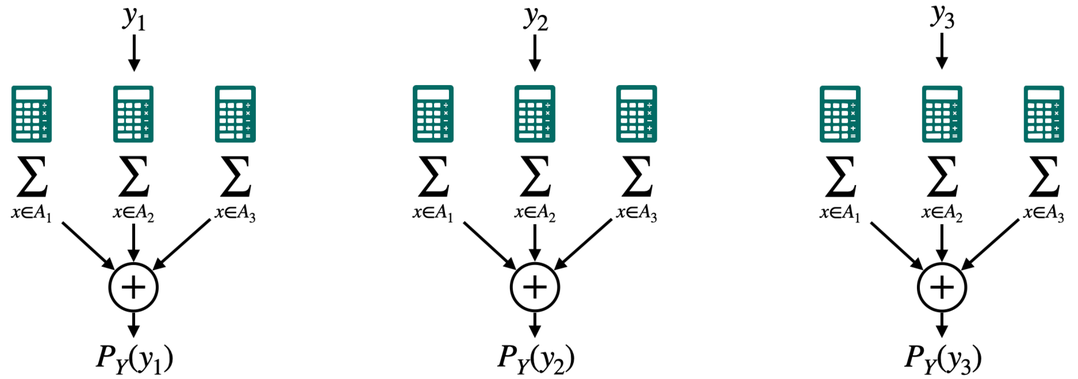

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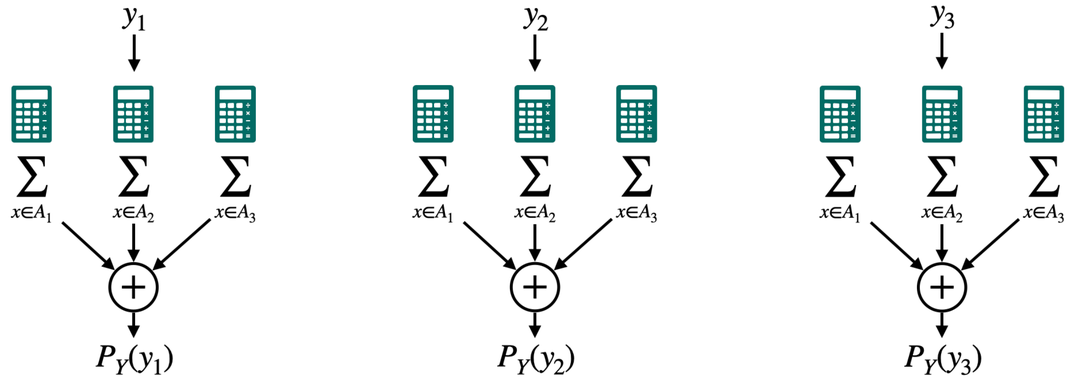

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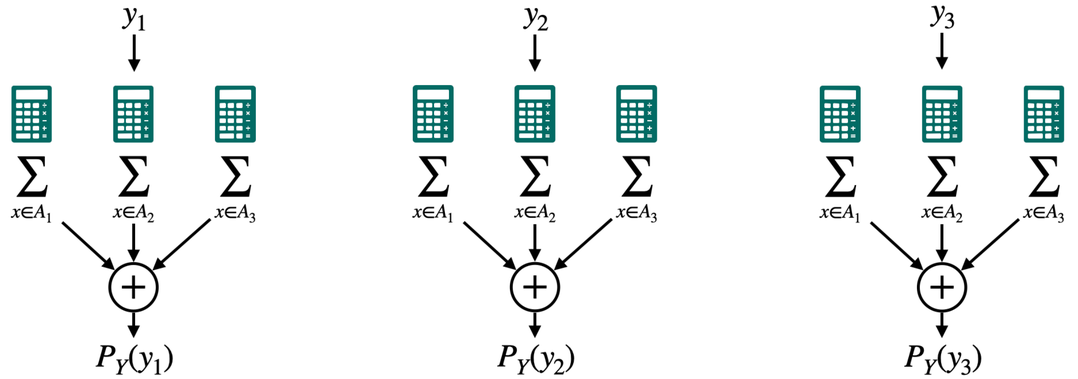

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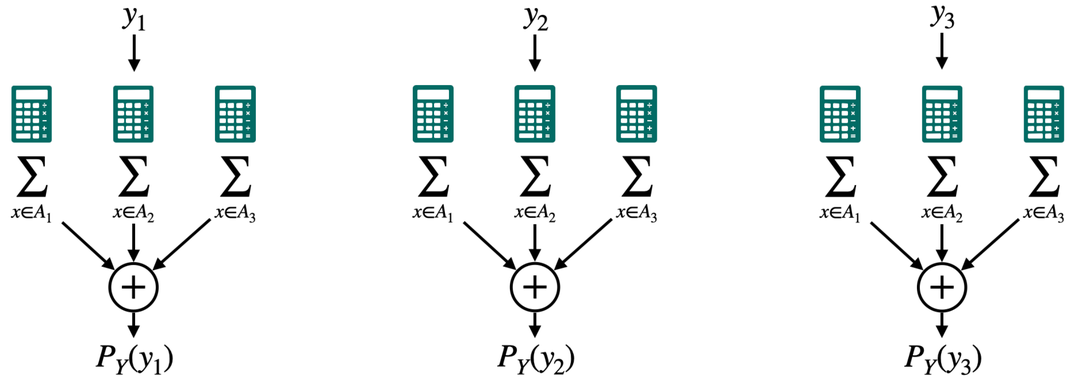

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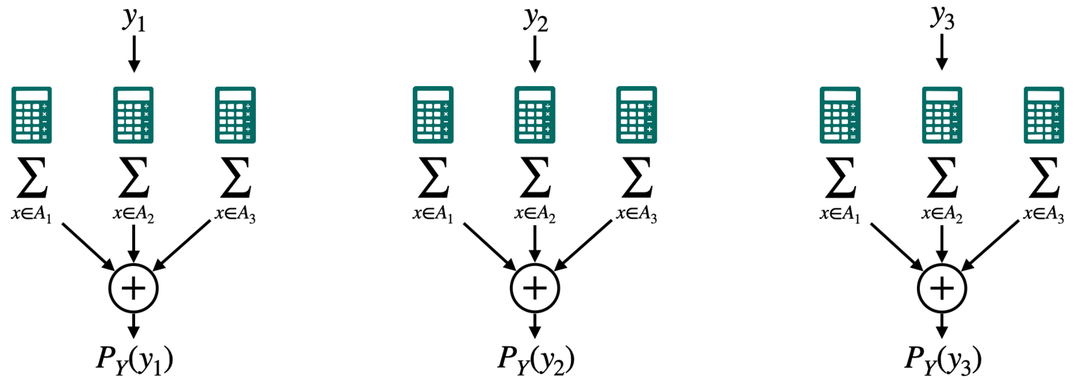

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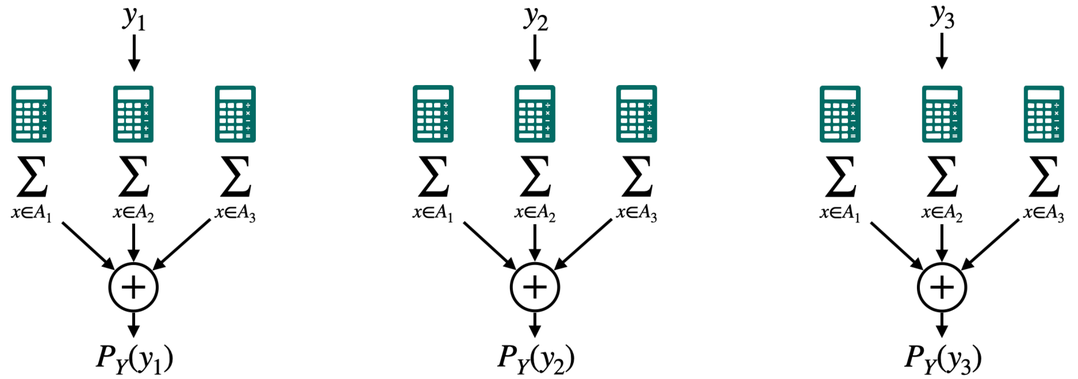

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- On the other hand, our parallel implementation can be applied more broadly. Where else could it be useful? Maybe you know. :)

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- We get improved upper bounds on the capacity of the deletion channel.

Wrapping up

- Understanding the capacity of the deletion channel is a major research direction;
- Capacity of deletion channel is upper bounded by capacity of any “fixed input length” deletion channel;
- In principle, can use Blahut-Arimoto to approximate these fixed-input-length capacities;
- But Blahut-Arimoto time/space complexity is exponential in input length L;
- We exploit simple properties of Blahut-Arimoto to parallelize it and fit it into GPUs;
- We use this parallelized version to approximate these fixed-input-length capacities for larger L;
- We get improved upper bounds on the capacity of the deletion channel.

Thanks!